MAT502 - Additional Problem Set 05

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1. The algebra of ***octonions*** (also called ***Cayley numbers*** is the 8-dimensional real vector space $\mathbb{O} = \mathbb{H} \times \mathbb{H}$ (where $\mathbb{H}$ is the space of quaternions) with the following bilinear product

$$(p, q)(r, s) = (pr - sq^*, p^*s + rq), \quad \text{for } p, q, r, s \in \mathbb{H}.$$

Show that $\mathbb{O}$ is a noncommutative, nonassociative algebra over $\mathbb{R}$, and prove that there exists a smooth global frame on S^7. [Hint: it might be helpful to prove that $(PQ^*)Q = P(Q^*Q)$ for all $P, Q \in \mathbb{O}$, where $(p, q)^* = (p^*, -q)$.

2. Show by finding a counterexample that Proposition 8.19 is false if we replace the assumption that F is a diffeomorphism by the weaker assumption that it is smooth and bijective.

3. For each of the following vector fields on the plane, compute its coordinate representation in polar coordinates on the right half-plane $\{(x, y) : x > 0\}$.

- **a.** $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$
- **b.** $Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$
- **c.** $Z = (x^2 + y^2)\frac{\partial}{\partial x}$

4. For each of the following pairs of vector fields X, Y defined on $\mathbb{R}^3$, compute the Lie bracket $[X, Y]$.

- **a.** $X = y\frac{\partial}{\partial z} - 2xy^2\frac{\partial}{\partial y}$; $Y = \frac{\partial}{\partial y}$
- **b.** $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$; $Y = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$

5. Prove the following properties of induced homomorphisms.

- **a.** The homomorphism $(\mathrm{Id}_G)_* : \mathrm{Lie}(G) \to \mathrm{Lie}(G)$ induced by the identity map of G is the identity map of $\mathrm{Lie}(G)$.
- **b.** If $F_1 : G \to H$ and $F_2 : H \to K$ are Lie group homomorphisms, then

$$(F_2 \circ F_1)_* = (F_2)_* \circ (F_1)_* : \mathrm{Lie}(G) \to \mathrm{Lie}(K).$$

- **c.** Isomorphic Lie groups have isomorphic Lie algebras.

6. Verify that the kernel and image of a Lie algebra homomorphism are Lie subalgebras.

7. Let $X, Y, Z \in \mathfrak{X}(M)$. Show that, for $f, g \in C^\infty(M)$, the Lie bracket satisfies

$$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X.$$

8. *[Spring 2015]* Let G be a Lie group which acts smoothly on the smooth manifolds M and N.

- **a.** Suppose that $\Phi : M \to N$ is a smooth map that is equivariant relative to the actions on M and N. Suppose further that the action of G on M is transitive. Prove that Φ has constant rank.
- **b.** Use part (a) to prove that $O(n)$ is a smooth $n(n-1)/2$-dimensional Lie subgroup of $GL(n, \mathbb{R})$. Describe the tangent space $T_I O(n) \subset M(n, \mathbb{R})$.
- **c.** Prove that $O(n)$ is compact