MAT502 - Additional Problem Set 04

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1. Our definition of Lie groups includes the requirement that both the multiplication map and the inversion map are smooth. Show that the smoothness of the inversion map is redundant: if G is a smooth manifold with a group structure such that the multiplication map $m : G \times G \to G$ is smooth, then G is a Lie group. [Hint: show that the map $F : G \times G \to G \times G$ defined by $F(g, h) = (g, gh)$ is a bijective local diffeomorphism.]

2. Determine which of the following Lie groups are compact:

$$\mathrm{GL}(n, \mathbb{R}),\ \mathrm{SL}(n, \mathbb{R}),\ \mathrm{GL}(n, \mathbb{C}),\ \mathrm{SL}(n, \mathbb{C}),\ U(n),\ SU(n).$$

3. If G is a Lie group and $F : G \to H$ is a Lie group, then its universal cover $\tilde{G}$ admits a Lie group structure. What other important detail can you add to this statement?

4. True or False? If G is a Lie group and $F : G \to H$ is a Lie group homomorphism, then $F(G)$ is an embedded Lie subgroup of H.

5. True or False? If $X \in \mathfrak{X}(M)$ satisfies $X(f) = 0$ for all $f \in C^\infty(M)$, then $X \equiv 0$.

6. If X is a smooth nontrivial vector field, is XX (i.e. $X \circ X$) *ever* a smooth vector field?

7. Suppose that M and N are smooth manifolds, $F : M \to N$ is a diffeomorphisms, and $X, Y \in \mathfrak{X}(M)$. Prove that $F_*[X, Y] = [F_*X, F_*Y]$. (Here $F_* : \mathfrak{X}(M) \to \mathfrak{X}(N)$ is the push-forward of vector-fields induced by F.)

8. Suppose $F : G \to H$ is a Lie group homomorphism and $X \in \mathrm{Lie}(G)$.

- **a.** Define F_*X. Does your definition require that F is actually an isomorphism?
- **b.** Prove that any $X \in \mathrm{Lie}(G)$ is complete.
- **c.** Show that $\mathrm{O}(n)$ is compact.

9. Let G and H be Lie groups and $F : G \to H$ a Lie group homomorphism.

- (a) Prove that F has constant rank.
- (b) Explain why part (a) implies that $\ker(F)$ is an embedded Lie subgroup of G.

10. Give an example of two non-isomorphic Lie groups with isomorphic Lie algebras.

11. Let $S = \{A \in M_2(\mathbb{R}) \ :\ \mathrm{rank}(A) = 1\}$, where $M_2(\mathbb{R})$ is the set of 2×2 matrices with real entries.

- (a) Prove that S is a 3-dimensional embedded submanifold of $M_2(\mathbb{R})$.
- (b) Consider the following matrix in S:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

  What is the tangent space $T_A S$ when regarded as a subspace of $M_2(\mathbb{R})$ via the identification $T_A M_2(\mathbb{R}) \cong M_2(\mathbb{R})$?