MAT502 - Additional Problem Set 01

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1. What is a smooth structure on a topological manifold? Do all topological manifolds admit smooth structures?
2. Is a smooth structure on a smooth manifold ever unique?
3. Show that $\mathbb{RP}^n$ is compact.
4. Let M be a topological manifold. Prove that two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.
5. Let N denote the ***north pole*** $(0, \ldots, 0, 1) \in \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$, and let S denote the ***south pole*** $(0, \ldots, 0, -1)$. Define the ***stereographic projection*** $\sigma : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$ by

$$\sigma(x^1, \ldots, x^{n+1}) = \frac{(x^1, \ldots, x^n)}{1 - x^{n+1}}.$$

Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}^n \setminus \{S\}$.

   a. For any $x \in \mathbb{S}^n \setminus \{N\}$, show that $\sigma(x) = u$, where $(u, 0)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$. Similarly, show that $\tilde{\sigma}(x)$ is the point where the line through S and x intersects the same subspace. (For this reason, $\tilde{\sigma}$ is called the ***stereographic projection from the south pole***.)

   b. Show that σ is bijective, and

$$\sigma^{-1}(u^1, \ldots, u^n) = \frac{(2u^1, \ldots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}.$$

   c. Compute the transition map $\tilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas consisting of the two charts $(\mathbb{S}^n \setminus \{N\}, \sigma)$ and $(\mathbb{S}^n \setminus \{S\}, \tilde{\sigma})$ defines a smooth structure on $\mathbb{S}^n$. (The coordinates are defined by σ or $\tilde{\sigma}$ are called the ***stereographic coordinates***.)

6. Let M be a smooth manifold with or without boundary. Show that pointwise multiplication turns $C^\infty(M)$ into a commutative ring and a commutative associate $\mathbb{R}$-algebra.
7. Suppose that M is a smooth manifold. Prove that M admits a smooth proper function $f : M \to \mathbb{R}$.
8. For any topological space M, let $C(M)$ denote the algebra of continuous functions $f : M \to \mathbb{R}$. Given a continuous map $F : M \to N$, define $F^* : C(N) \to C(M)$ by $F^*(f) = f \circ F$.

   a. Show that F^* is a linear map.

   b. Suppose M and N are smooth manifolds. Show that $F : M \to N$ is smooth if and only if $F^*(C^\infty(N)) \subseteq C^\infty(M)$.

   c. Suppose $F : M \to N$ is a homeomorphism between smooth manifolds. Show that it is a diffeomorphism if and only if F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.