GEOMETRY AND TOPOLOGY QUALIFYING EXAM: SPRING 2015

Problem 1: Let S be the topological space with the polygonal presentation

$$\langle a, b, c, d, e | aba^{-1}cb^{-1}ded^{-1}ec^{-1} \rangle.$$

(a) Is S a manifold?

(b) Is S orientable?

(c) What is the Euler characteristic $\chi(S)$?

(d) To which standard surface is S homeomorphic?

Problem 2:

(a) Let X denote the topological space obtained by gluing the meridian of a torus to a longitude of another torus. Find a presentation for $\pi_1(X)$.

(b) Prove that the Möbius band does not retract onto its boundary circle.

Problem 3:

(a) Prove or disprove: in any topological space, the product of paths is associative.

(b) Let X be a path-connected, locally path-connected topological space with $\pi_1(X)$ finite. Prove that any continuous map $f : X \longrightarrow S^1$ is nullhomotopic.

Problem 4: Let F_n denote the free group on n generators. Prove that for any $n \geqslant 3$, F_2 contains a subgroup isomorphic to F_n. What is the index of this subgroup?

Problem 5. Let M be a smooth manifold and $f \in C^\infty(M)$.

(a) Give the definition of the differential df of f.

(b) Suppose that M is connected. Prove that f is constant on M if and only if $df = 0$.

(c) State and prove the "fundamental theorem" for line integrals.[1]

Problem 6. Let G be a Lie group which acts smoothly on the smooth manifolds M and N.

(a) Suppose that $\Phi : M \to N$ is a smooth map that is equivariant relative to the actions on M and N. Suppose further that the action of G on M is transitive. Prove that Φ has constant rank.[2]

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[1] This is the theorem which expresses the value of $\int_\gamma df$ for a smooth function f and piecewise smooth path γ.

[2] *Hint: Mimic the proof that a Lie group homomorphism has constant rank.*

(b) Use (a) to prove that $O(n)$ is a smooth $n(n-1)/2$-dimensional Lie subgroup of $GL(n,\mathbb{R})$. Describe the tangent space $T_I O(n) \subset M(n,\mathbb{R})$. [3]

(c) Prove that $O(n)$ is compact.

Problem 7. Let M be a smooth compact oriented manifold with boundary ∂M.

(a) Prove that there does not exist a smooth retraction of $r : M \to \partial M$.[4]

(b) Let B be the closed unit ball in $\mathbb{R}^n$. Use (a) to show that every smooth map $F : B \to B$ must have a fixed point.[5]

Problem 8. Let (M, g) be a smooth n-dimensional Riemannian manifold.

(a) Let N be a smooth manifold and $F : N \to M$ be a smooth map. Prove that $h = F^*g$ is a smooth metric on N if and only if F is an immersion.

(b) Part (a), together with a certain big theorem, implies that every smooth manifold admits a Riemannian metric. State (some version of) this big theorem.

(c) Use the existence of smooth partitions of unity to give another proof that every manifold admits a Riemannian metric.

(d) Prove that if $n = 1$, (M, g) is locally isometric to $(\mathbb{R}, g_{\text{euc}})$.[6]

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[3] *Hint: For the first part, let $GL(n,\mathbb{R})$ act on itself by right multiplication and define a right action of $GL(n,\mathbb{R})$ on $M(n,\mathbb{R})$ by $X \cdot B = B^T X B$. Then consider the map $\Phi : GL(n,\mathbb{R}) \to M(n,\mathbb{R})$ given by $\Phi(A) = A^T A$.*

[4] *Hint: Consider an orientation form $\omega_{\partial M}$ on ∂M and use Stokes's theorem.*

[5] *Hint: Any map from B to B without a fixed point can be used to define an explicit retraction $\rho : B \to \partial B$.*

[6] *Hint: Use part (a). The expression of g in any local coordinate t will have the form $g = f(t)dt \otimes dt$ for some smooth positive f.*