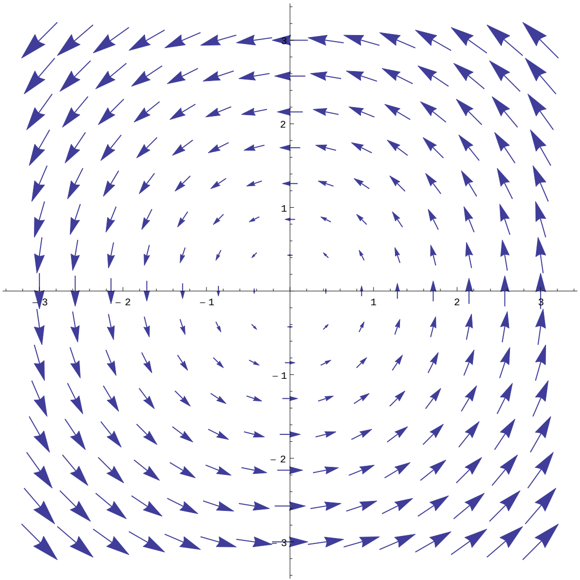
Recitation 13: Vector Fields, Line Integrals, Conservative Vector Fields

Joseph Wells
Arizona State University

April 24, 2015

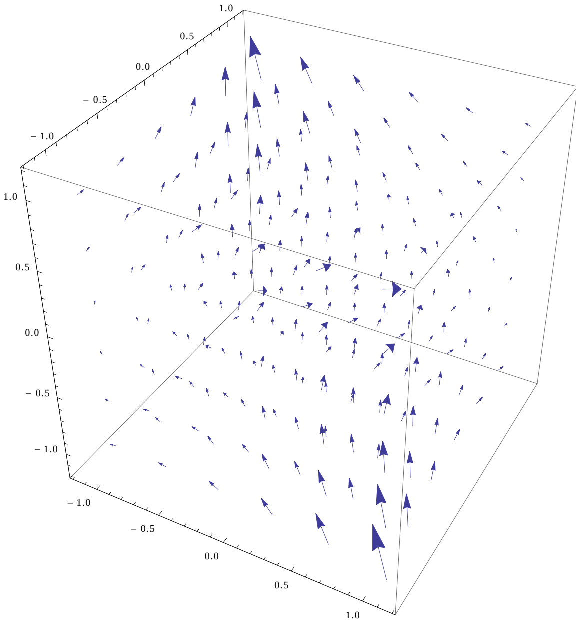
Formally, a **vector field** is a map that takes a point in space to a vector in space. The way we often think about these is a surface covered in arrows, where the arrows emanate from the corresponding point.

Example. Sketch the vector field $\mathbf{F} = \mathbf{F}(x, y) = \langle x, -y \rangle$.



There are two important types of vector fields. The first are **radial vector fields**, which have the form $\mathbf{F}(x, y) = f(x, y)\langle x, y\rangle$. The second are **gradient vector fields**. For a given real-valued function $\varphi(x, y)$, the associated gradient field is $\mathbf{F}(x, y) = \nabla\varphi(x, y)$. We call φ the **(scalar) potential function** for F.

Example. For $\varphi(x, y, z) = ze^{-xy}$, find and sketch $F = \nabla\varphi$.



This procedure makes sense because the gradient of a scalar function is a vector.

$$
\begin{aligned}
F(x, y, z) &= \nabla\varphi(x, y, z) \\
&= \nabla\left\langle -yze^{-xy}, -xze^{-xy}, e^{-xy} \right\rangle .
\end{aligned}
$$

Suppose $z = f(x, y)$ is a surface in $\mathbb{R}^3$ and C is a parametrized curve in the xy-plane. If $C = \mathbf{r}(s) = \langle x(s), y(s)\rangle$ is parametrized by arc length, where $a \leq s \leq b$, the **line integral over** C is

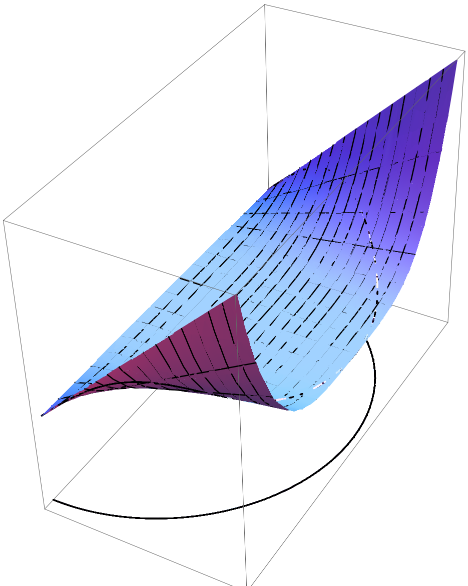
$$\int_C f\, ds = \int_a^b f \circ C\, ds = \int_a^b f(x(s), y(s))\, ds.$$

If $C = \mathbf{r}(t) = \langle x(t), y(t)\rangle$ is an arbitrary parametrization, where $a \leq t \leq b$, then the line integral is

$$\int_C f\, ds = \int_a^b f \circ C|\mathbf{r}'(t)|\, dt = \int_a^b f(x(t), y(t))\, |\mathbf{r}'(t)|\, dt.$$

Of course, this latter integral agrees with the arc length parametrization case.

Example. Evaluate $\int_C xy^4\,ds$, where C is the right half of the circle $x^2+y^2=16$, traveling counter-clockwise.



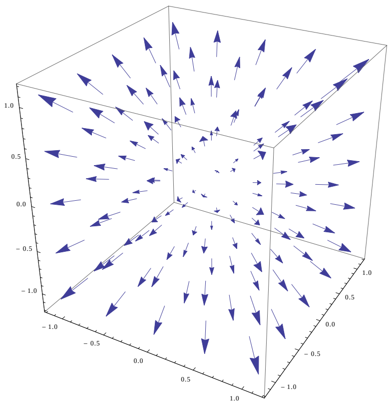
We can parametrize C via $C = \mathbf{r}(t) = \langle 4\cos(t), 4\sin(t)\rangle$, where $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. We then have that $|\mathbf{r}'(t)| = \sqrt{16\sin^2(t)+16\cos^2(t)} = 4$, so the integral is

$$\begin{aligned}\int_C f\,ds &= \int_{-\pi/2}^{\pi/2} f(4\cos(t), 4\sin(t))\,4\,dt \\ &= 4096\int_{-\pi/2}^{\pi/2}\cos(t)\sin^4(t)\,dt \\ &= \frac{8192}{5}.\end{aligned}$$

Suppose F is a continuous vector field defined on a smooth parametrized curve $C = \mathbf{r}(\mathbf{t}) = \langle x(t), y(t), z(t)\rangle$. Let $\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ be the unit tangent vector for $\mathbf{r}(t)$. Then the **line integral of F on** C is given by

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}(x(t), y(t), z(t))\cdot\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}|\mathbf{r}'(t)|\,dt = \int_a^b \mathbf{F}(x(t), y(t), z(t))\cdot\mathbf{r}'(t)\,dt. = \int_C \mathbf{F}\,\cdot$$

Example. Evaluate $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$, where $\mathbf{F}(x,y,z) = \langle xz, 0, -yz\rangle$ and C is the line segment from $(-1,2,0)$ to $(3,0,1)$.



We can parametrize C via $C = \mathbf{r}(t) = \langle 4t-1, 2-2t, t\rangle$, where $0 \le t \le 1$. We then have that $\mathbf{r}'(t) = \langle 4, -2, 1\rangle$. Thus

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{T}\,ds &= \int_0^1 F(4t-1, 2-2t, t)\cdot\mathbf{r}'(t)\,dt\\ &= \int_0^1 \langle 4t^2 - t, 0, 2t^2 - 2t\rangle\cdot\langle 4, -2, 1\rangle\,dt\\ &= \int_0^1 18t^2 - 6t\,dt\\ &= 3.\end{aligned}$$

An important application of integrating vector fields is to calculate work done (as gravity can be thought of as a vector field). In particular,

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Example. Find the work done by a person weighing 150 pounds walking exactly one revolution up a circular helical staircase of radius 3 feet, if the person rises 10 feet.

Gravity is constantly pulling down, so all work is done in the vertical direction. As such, we would like work to be positive when ascending the stairs and fighting gravity, so we write our force field as $\mathbf{F} = \langle 0, 0, 150\rangle$. We can also parametrize the helical staircase as $C = \mathbf{r}(t) = \langle 3\cos(2\pi t), 3\sin(2\pi t), 10t\rangle$, where $0 \leq t \leq 1$. Then $\mathbf{r}'(t) = \langle -6\pi\sin(2\pi t), 6\pi\cos(2\pi t), 10\rangle$, and the amount of work done is

$$\begin{aligned} W &= \int_0^1 \mathbf{F}(3\cos(2\pi t), 3\sin(2\pi t), 10t) \cdot \mathbf{r}'(t)\, dt \\ &= \int_0^1 \langle 0, 0, 150\rangle \cdot \langle -6\pi\sin(2\pi t), 6\pi\cos(2\pi t), 10\rangle\, dt \\ &= \int_0^1 1500\, dt = 1500\,\mathrm{ft} - \mathrm{lb}. \end{aligned}$$

A vector field $\mathbf{F}$ is **conservative** if there exists a real-valued function φ such that $F = \nabla\varphi$. In general, vector fields need not arise this way. But we do have some necessary and sufficient conditions for when they do.

Theorem 1. *Let $\mathbf{F} = \langle f, g \rangle$ be a vector field on a connected, simply-connected region in $\mathbb{R}^2$. Then F is conservative if and only if*

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}.$$

Example. Let $\mathbf{F}(x,y) = \langle 2x^3y^4 + x, 2x^4y^3 + y\rangle$ be a vector field in $\mathbb{R}^2$. Is it conservative? If so, find the potential function φ for $\mathbf{F}$.

Let $f(x,y) = 2x^3y^4 + x$ and $g(x,y) = 2x^4y^3 + y$. Then we have that $\frac{\partial f}{\partial y} = 8x^3y^3 = \frac{\partial g}{\partial x}$, so $\mathbf{F}$ is conservative. So, let φ be the potential function. We know that $\varphi_x = 2x^3y^4 + x$, so

$$\varphi(x,y) = \int \varphi_x \, dx = \int 2x^3y^4 + x \, dx = \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + G(y).$$

Now, we know that $2x^4y^3 + y = \varphi_y = 2x^4y^3 + G'(y)$, so in fact we must have that $G'(y) = y$. Hence

$$G(y) = \int G'(y)\, dy = \int y \, dy = \frac{1}{2}y^2 + C.$$

Putting it all together, we get that our potential function is of the form:

$$\varphi(x,y) = \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + C.$$

Assignment

Worksheet 13:

`https://mathpost.asu.edu/~wells/math/teaching/mat272_spring2015/homework13.pdf`

As always, you may work in groups, but every member must individually submit a homework assignment.