Recitation 09: Lagrange Multipliers & Integration over Regions

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Lagrange multipliers are just constants that crop up in optimization problems.

Procedure:

Let f be your objective function (the one we're trying to optimize) and g the the constraint function. Suppose both f and g are differentiable on a region of $\mathbb{R}^2$ with $\nabla g(x,y) \neq \overrightarrow{0}$ on the curve $g(x,y)=0$. To locate the maximum and minimum values of f subject to the constraint $g(x,y)=0$,

1. Find values of x, y, and λ (if they exist) that satisfy the equations

$$\nabla f(x,y) = \lambda \nabla g(x,y) \quad \text{and} \quad g(x,y) = \overrightarrow{0}.$$

2. For all pairs (x,y) from step 1, choose the ones corresponding to the largest and smallest values of $f(x,y)$.

In other words, step one is just solving the system of equations:

$$f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad \text{and} \quad g(x,y) = 0.$$

Note: for more dimensions, this procedure is completely analogous.

Example. Find the maximum and minimum values of $f(x, y, z) = 2x + z^2$ subject to the constraint that $x^2 + y^2 + 2z^2 = 25$.

Solution.

Our constraint function is $g(x, y, z) = x^2 + y^2 + 2z^2 - 25$. $\nabla f(x, y, z) = \langle 2, 0, 2z\rangle$ and $\nabla g(x, y, z) = \langle 2x, 2y, 4z\rangle$. Since $\nabla g(x, y, z) = 0$ precisely when $(x, y, z) = (0, 0, 0)$ (which is not in the domain of g), there exists some real number λ we solve the system of equations:

$$\begin{aligned} 2 &= \lambda 2x \\ 0 &= \lambda 2y \\ 2z &= \lambda 4z \\ x^2 + y^2 + 2z^2 &= 25. \end{aligned}$$

The first equation implies that $x = \frac{1}{\lambda}$, so $\lambda \neq 0$.
The second equation implies that $\lambda = 0$, which cannot happen, or $y = 0$. So $y = 0$.
The third equation implies that either $z = 0$ or $\lambda = \frac{1}{2}$.

If $z = 0$, then $x^2 - 25 = 0$ implies that $x = \pm 5$, hence $(5, 0, 0)$ and $(-5, 0, 0)$ are two possible points for the max/min.

If $\lambda = \frac{1}{2}$, then $x = 2$, so $4 + z^2 - 25 = 0$ implies that $z = \pm\sqrt{21}$, so $(2, 0, -\sqrt{21})$ and $(2, 0, \sqrt{21})$ are other possible candidates. Checking these explicitly,

$$\begin{aligned} f(-5, 0, 0) &= -10, \\ f(5, 0, 0) &= 10, \\ f(2, 0, -\sqrt{21}) &= 4 - 21 = -17, \\ f(2, 0, \sqrt{21}) &= 4 + 21 = 25. \end{aligned}$$

Thus $(2, 0, -\sqrt{21})$ corresponds to the minimum and $(2, 0, \sqrt{21})$ corresponds to the maximum.

Definition. Let $f(x, y)$ be a function defined on a rectangular region R in the xy-plane. f is **integrable** if $\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$ exists for all partitions of R. This limit is the **double integral over** R, which we write

$$\iint_R f(x, y)\, dA = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k.$$

This integral is the volume of the solid bounded above by the surface $z = f(x, y)$ and the xy-plane over the region R.

In the real world, these Riemann sums are absolutely horrible to work with.

Let $z = f(x, y)$ be a surface and let $R := \{(a, b) : a \leq x \leq b, c \leq y \leq d\}$ be a region. Holding y fixed at some constant, we ultimately end up with just a 2-D curve, and we can take the integral under that curve to get the area $A(y) = \int_a^b f(x, y)\, dx$. Now we have a function of the area under each slice of the curve. Integrating over those gives us the volume $V = \int_c^d A(y)\, dy$.

In this class (or at least, this section anyway), the order of integration is unimportant, and it is formally stated as the following theorem:

Theorem 1 (Fubini(-Tonelli)). *If* $f(x,y)$ *is continous and* $R = \{(x,y) \, : \, a \le x \le b, \, c \le y \le d\}$ *is a rectangular region,*

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

Example. Evaluate $\displaystyle\iint_R (x^2 - y^2)^2 \, dA$ over the region $R = \{(x, y) : -1 \le x \le 2,\ 0 \le y \le 1\}$.

Solution.

$$\begin{aligned}
\int_0^1 \int_{-1}^2 (x^2 - y^2)^2 \, dx \, dy &= \int_0^1 \int_{-1}^2 \left(x^4 - 2x^2y^2 + y^2\right) \, dx \, dy \\
&= \int_0^1 \left[\frac{1}{5}x^5 - \frac{2}{3}x^3y^2 + y^2x\right]_{-1}^{2} \, dy \\
&= \int_0^1 \left(3y^4 - 6y^2 + \frac{33}{5}\right) \, dy \\
&= \left[\frac{3}{5}y^5 - 2y^3 + \frac{33}{5}y\right]_0^1 = \frac{26}{5}.
\end{aligned}$$

As it turns out, the whole reason for Fubini's theorem is that it's sometimes much easier to evaluate the integral in one order than the other. This is demonstrated by the following example.

Example. Evaluate $\iint_R x\sec^2(xy)\,dA$ over the region $R = \{(x,y) : 0 \le x \le \frac{\pi}{3}, 0 \le y \le 1\}$.

Solution. We'll first try

$$\int_0^1 \int_0^{\pi/3} x\sec^2(xy)\,dx\,dy.$$

Remember, here y is just a constant. To do this, we'll have to integrate by parts with $u = x$ and $dv = \sec^2(xy)\,dx$. Then $du = dx$ and $v = \frac{1}{y}\tan(xy)$. And...eww.

Now try

$$\int_0^{\pi/3} \int_0^1 x\sec^2(xy)\,dy\,dx = \int_0^{\pi} x \int_0^1 \sec^2(xy)\,dy\,dx$$

$$
\begin{aligned}
&= \int_0^{\pi/3} x \left[\frac{1}{x} \tan(xy) \right]_{y=0}^{y=1} dx \\
&= \int_0^{\pi/3} x \left[\frac{1}{x} \tan(x) - \frac{1}{x} \tan(0) \right] dx \\
&= \int_0^{\pi/3} \tan(x)\, dx \\
&= \big[-\ln|\cos(x)| \big]_0^{\pi/3} \\
&= \ln(2).
\end{aligned}
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Assignment

Worksheet 09:

`https://mathpost.asu.edu/~wells/math/teaching/mat272_spring2015/homework09.pdf`

As always, you may work in groups, but every member must individually submit a homework assignment.