Recitation 08: Tangent Planes & Linear Approximation; Maximum/Minimum Problems

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March 16, 2015

Since the gradient at a point is a vector normal to the surface at that point, that normal vector uniquely defines a plane tangent to the surface at that point.

Definition. Let $F(x, y, z) = 0$ be some surface and $P = (a, b, c)$ a point on the surface where $\nabla F(a, b, c) \neq 0$. The **tangent plane** to the surface at P is given by the equation

$$\nabla F(a, b, c) \cdot \langle x - a, y - b, z - c \rangle = 0$$

which expands to

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$$

Definition. With F and P as above, the **normal line** to $F(x, y, z) = 0$ at P is given by

$$r(t) = \langle x(t), y(t), z(t) \rangle = \overrightarrow{P} + t\nabla F(a, b, c)$$

or, as set of parametric equations, by

$$\begin{cases} x(t) = a + tF_x(a, b, c) \\ y(t) = b + tF_y(a, b, c) \\ z(t) = c + tF_z(a, b, c) \end{cases}$$

Example. Find the equation of the tangent plane and normal line to the surface $z = e^{xy}$ at the point $(1, 0, 1)$.

Solution.

Notice that for $F(x, y, z) = e^{xy} - z$, this suface is exactly $F(x, y, z) = 0$. We then have that

$$\nabla F(x, y, z) = \langle ye^{xy}, xe^{xy}, -1\rangle,$$
$$\nabla F(1, 0, 1) = \langle 0, 1, -1\rangle,$$

so the equation of the tangent plane is

$$\nabla F(1, 0, 1) \cdot \langle x - 1, y - 0, z - 1\rangle = y - (z - 1) = 0.$$

The normal line to F at the point $(1, 0, 1)$ is given by

$$r(t) = \langle 1, 0, 1\rangle + t\nabla F(1, 0, 1) = \langle 1, t, 1 - t\rangle.$$

Definition. Let f be differentiable at (a, b). The **linear approximation to the surface** $z = f(x, y)$ **at the point** $(a, b, f(a, b))$ is the tangent plane at that point, given by

$$L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

Example. Compute the linear approximation of the function $f(x, y) = \ln(1 + x + y)$ at the point $(0, 0)$. Use this to estimate the value $f(0.1, -0.2)$.

Solution.

$$\begin{aligned} L(x, y) &= f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0) + f(0, 0) \\ &= \frac{1}{1 + 0 + 0}(x - 0) + \frac{1}{1 + 0 + 0}(y - 0) + \ln(1 + 0 + 0) \\ &= x + y \end{aligned}$$

so

$$f(0.1, -0.2) = -0.105361 \ldots \approx -0.1 = L(0.1, -0.2).$$

Definition. Let f be differentiable at the point (a, b). For $z = f(x, y)$, the **differential** dz is the change from $f(a, b)$ to $f(a + dx, b + dy)$ and is given by

$$dz = \nabla f \cdot \langle dx, dy \rangle = f_x(a, b)\, dx + f_y(a, b)\, dy.$$

dz is sometimes called the *total differential.*

Example. Find the total differential of the function $w = f(u, x, y, z) = \dfrac{u + x}{y + z}$.

Solution

$$dw = \frac{1}{y + z}\, du + \frac{1}{y + z}\, dx - \frac{u + x}{(y + z)^2}\, dy - \frac{u + x}{(y + z)^2}\, dz$$

Definition. Let f be a function differentiable at (a, b). We say that (a, b) is a **local maximum** if every point (x, y) in a neighborhood of (a, b) has the property that $f(x, y) \leq f(a, b)$. Similarly, (a, b) is a **local minimum** if every point (x, y) in a neighborhood of (a, b) has the property that $f(x, y) \geq f(a, b)$.

Theorem 1. *If f has a local max or local min at (a, b) and f_x, f_y exist, then $f_x(a, b) = f_y(a, b) = 0$.*

Definition. A point (a, b) is called a **critical point** of f if $f_x(a, b) = f_y(a, b) = 0$ or if at least one of f_x or f_y does not exist.

Definition. A critical point (a, b) of a function f is a **saddle point** if there exist points (x, y) in a neighborhood of (a, b) such that $f(x, y) > f(a, b)$ and $f(x, y) < f(a, b)$.

Theorem 2 (Second Derivative Test). *Suppose that second partial derivatives of f are continuous in a neighborhood of (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Let $D(x, y) = f_{xx}f_{yy} - f_{xy}^2$, which we call the* ***discriminant of*** *f.*

1. *If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b).*
2. *If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b).*
3. *If $D(a, b) < 0$, then f has a saddle point at (a, b).*
4. *If $D(a, b) = 0$, then the test is inconclusive.*

Example. Locate and classify all critical points of $f(x, y) = x^4 + 2y^2 - 4xy$.

Solution.

First we need to find the critical points. We require that $f_x = f_y = 0$, so we solve

$$f_x = 4x^3 - 4y = 0$$
$$f_y = 4y - 4x = 0$$

to get that we have critical points at $(0, 0)$, $(-1, -1)$, and $(1, 1)$.

Via the second derivative test, we see that $D(0, 0) < 0$, so $(0, 0)$ is a saddle point. Also, $D(-1, -1), D(1, 1) > 0$ and $f_{xx}(-1, -1), f_{xx}(1, 1) > 0$, so $(-1, -1)$ and $(1, 1)$ are local minima.

Assignment

Worksheet 08:

https://mathpost.asu.edu/~wells/math/teaching/mat272_spring2015/homework08.pdf

As always, you may work in groups, but every member must individually submit a homework assignment.