Recitation 07: Partial Derivatives; The Chain Rule; Directional Derivatives & The Gradient

Joseph Wells
Arizona State University

March 2, 2015

Given a multivariable function, we saw that limits were no longer as straightforward as the single-variable case because we had to consider different paths. Since the derivative is defined as a limit, it means that differentiating takes on a set of challenges too. So what's the point of the partial derivative? The partial derivative is preformed by fixing a curve along our surface and differentiating with respect to that curve in the usual sense. In practice, we'll simply fix a curve in the x- (resp. y-direction), which allows us to treat y (resp. x) as a constant and just differentiate with respect to x (resp. y).

To denote a partial derivative, we use a kind of curvy lowercase "d", which you may hear called "partial" (or just "d" if the context is clear; afterall, our usual derivative is just a partial derivative of a function of one variable), or with subscripts

$$\frac{\partial f}{\partial x} \quad \text{or} \quad f_x.$$

Example. Compute the first partial derivatives, $\frac{\partial f}{\partial x} = f_x$ and $\frac{\partial f}{\partial y} = f_y$ of the function $f(x, y) = x^{2y}$.

Solution.

Treating y as a constant and differentiating with respect to x, we get

$$f_x = 2yx^{2y-1}.$$

We first recall from Calc I that $\frac{d}{dt}[a^t] = a^t \ln(a)$, so treating x as a constant and differentiating with respect to y, we get

$$f_y = 2x^{2y} \ln(x).$$

We can also take higher-order partial derivatives as well, although since we now have two choices for variables with respect to which we can differentiate, so we can have mixed partial derivatives. In particular, given a function $f(x, y)$, the second-order partial derivatives are

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(f)\right) = (f_x)_x = f_{xx}$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}(f)\right) = (f_x)_y = f_{xy}$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}(f)\right) = (f_y)_x = f_{yx}$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}(f)\right) = (f_y)_y = f_{yy}$$

Although it looks somewhat confusing with the letters swapping, you can think about the partial derivative as a sort of function (formally called an "operator") that acts on functions via composition.

Alas, the order isn't a huge deal, however. In this class, we will only concern ourselves with functions for which the order is irrelevant; i.e. $f_{xy} = f_{yx}$.

Example. Compute the four second partial derivatives for the function $Q(r,s) = \frac{r}{s}$.

Solution.

We first compute the first partial derivatives:

$$Q_r = \frac{1}{s} \qquad\qquad Q_s = -\frac{r}{s^2}.$$

From here, we compute the second partial derivatives:

$$\begin{aligned} Q_{rr} &= (Q_r)_r = 0 \\ Q_{rs} &= (Q_r)_s = -\frac{1}{s^2} \\ Q_{sr} &= (Q_s)_r = -\frac{1}{s^2} \\ Q_{ss} &= (Q_s)_s = \frac{2}{s^3}. \end{aligned}$$

Suppose $x(t)$, $y(t)$ are each functions of one variable and $f(x, y) = f(x(t), y(t))$ is a multivarabile function. Then the chain rule for ffunctions of one variable is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Example. Find $\frac{dz}{dt}$ where $z = x\sin(y)$, $x = t^2$, and $y = 4t^3$.

Solution.

Following from the chain rule, we need to calculate some various derivatives

$$\frac{\partial z}{\partial x} = \sin(y) \qquad \frac{dx}{dt} = 2t$$
$$\frac{\partial z}{\partial y} = x\cos(y) \qquad \frac{dy}{dt} = 12t^2.$$

Putting it all together, we get

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= 2t\sin(y) + 12t^2 x\cos(y) \\ &= 2t\sin(4t^3) + 12t^4\cos(4t^3).\end{aligned}$$

Suppose $z = f(x, y)$ is a function where $x = g(u, v)$ and $y = h(u, v)$ are also functions of multiple variables. Then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u},$$

and similarly for $\frac{\partial z}{\partial v}$.

Example. Find z_s and z_t where $z = x^2 \sin(y)$, $x = s - t$, and $y = t^2$.

Solution.

$$\frac{\partial z}{\partial x} = 2x\sin(y), \qquad \frac{\partial x}{\partial s} = 1, \qquad \frac{\partial y}{\partial s} = 0,$$
$$\frac{\partial z}{\partial y} = x^2\cos(y), \qquad \frac{\partial x}{\partial t} = -1, \qquad \frac{\partial y}{\partial t} = 2t.$$

So by the chain rule,

$$z_s = \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = 2x\sin(y) = 2(s-t)\sin(t^2),$$

$$\begin{aligned} z_t = \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} &= -2x\sin(y) + 2tx^2\cos(y) \\ &= -2(s-t)\sin(t^2) + 2t(s-t)^2\cos(t^2). \end{aligned}$$

Let f be differentiable at the point (x, y), the *gradient of* f, denoted ∇f or grad f, is the vector-valued function

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y).$$

It is important to note that the gradient output *is a vector*, and the vector is normal to the surface at the point, so it defines the tangent plane at that point.

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The *directional derivative of* f *at* (x, y) *in the direction of* $\mathbf{u}$ is

$$D_u f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = \langle f_x(x, y), f_y(x, y) \rangle \cdot \langle u_1, u_2 \rangle.$$

Example. Compute the gradient and the directional derivative of the given function at the given point and in the given direction: $f(x, y) = e^{-x-y}$, $P(\ln 2, \ln 3)$, $\mathbf{u} = \langle 1, 1 \rangle$

Let f be differentiable at (a, b).

1. f has the maximum rate of increase at (a, b) in the direction of $\nabla f(a, b)$. Rate is $|\nabla f(a, b)|$.

2. f has the maximum rate of decrease at (a, b) in the direction of $-\nabla f(a, b)$. Rate is $-|\nabla f(a, b)|$.

3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b)$.

Example.

Assignment

Worksheet 07:

`https://mathpost.asu.edu/~wells/math/teaching/mat272_spring2015/homework07.pdf`

As always, you may work in groups, but every member must individually submit a homework assignment.