Recitation 06: Graphs and Level Surfaces; Limits and Continuity

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Just as with functions of single variables, the *domain* for a multi-variable is the input set for which your function can be defined.

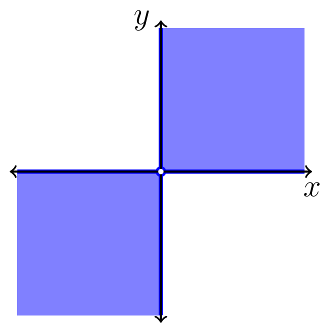
Example. Find the domain of the function $g(x, y) = \sqrt{\dfrac{xy}{x^2 + y^2}}$.

Solution

This is a square root, so we know that the radicand cannot be negative, and this only happens when either $xy < 0$. The square root is also undefined for the case where $x^2 + y^2 = 0$, which happens precisely when $x = y = 0$. So how can we write this as a domain, D? There are a couple of equivalent statements, but this one should work just fine:

$$D = \{(x, y) \in \mathbb{R}^2 \, : \, xy \geq 0 \text{ and } x^2 + y^2 \neq 0\}$$

Visually, our domain is the first and third quadrants of the plane (including the axes), minus the origin:



Functions of multiple variables can be represented as surfaces in $\mathbb{R}^3$, but those can sometimes be a bit hard to wrap our minds around. Instead, we can think about various curves representing elevations of a surface. This is a *level set*, and it is formally the set of all points (x, y) so that $f(x, y) = L$ for some fixed value L.

Example. Sketch three level curves for the function $f(x, y) = 3\cos(2x + y)$.

Solution

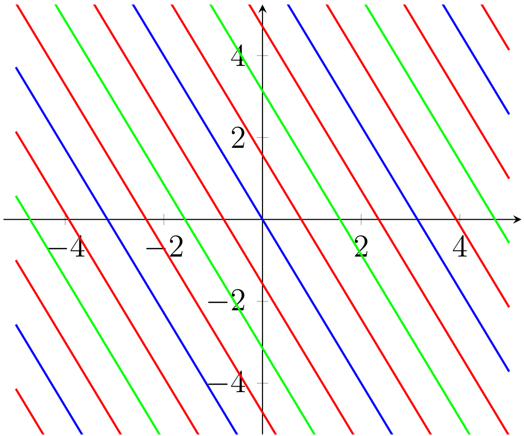
Set $z = f(x, y) = \cos(2x + y)$ and we'll choose three different levels to draw. How about the levels $z = 1$, $z = \frac{\sqrt{2}}{2}$, and $z = 0$.

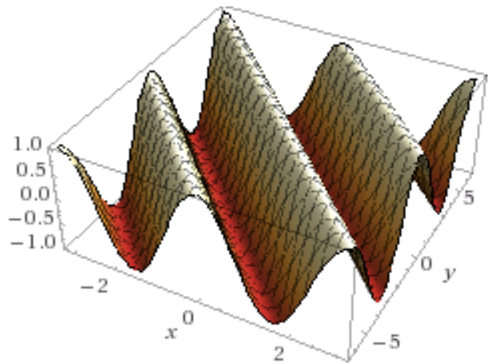
When $z = 1$, we have that $\cos(2x + y) = 1$, so $2x + y = 2n\pi$ and thus $y = -2x + 2n\pi$. The level set here is just diagonal lines.

When $z = 0$, we have that $\cos(2x + y) = 0$, so $2x + y = \left(2n + \frac{1}{2}\right)\pi$ or $\left(2n - \frac{1}{2}\right)\pi$ and thus $y = -2x + (2n + \frac{1}{2})\pi$, and $y = -2x + (2n - \frac{1}{2})\pi$. The level set here is just diagonal lines.

When $z = -1$, we have that $\cos(2x + y) = -1$, so $2x + y = (2n + 1)\pi$ and thus $y = -2x + (2n + 1)\pi$. The level set here is just diagonal lines.

Graphically, here are the level sets and the corresponding 3-D plot:





Limits of multi-variable functions are a natural analog to their single-variable counterparts with one subtle (but important) difference. In the single variable case, the limit exists if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$. Along a curve, there are only two paths you can take toward the limit point. In the multi-variable case, we have infinitely many (in fact, uncountably many) paths one could take to get to the limit point, and every single one of these must agree. Herein lies the power of the epsilon-delta definitions, as we can just take disks (or balls) around our point and argue that way, but I digress... In this class, we will not be too concerned with proving a limit exists, but merely finding the limit or stating that it does not exist.

Limit Rules:

Example. Compute the limit: $\lim\limits_{(x,y)\to(2,2)} \dfrac{y^2-4}{xy-2x}$.

Solution.

Using some factoring trickery from Calc I, we get

$$\begin{aligned}
\lim_{(x,y)\to(2,2)} \frac{y^2-4}{xy-2x} &= \lim_{(x,y)\to(2,2)} \frac{(y-2)(y+2)}{x(y-2)} \\
&= \lim_{(x,y)\to(2,2)} \frac{(y+2)}{x} \\
&= \frac{(2+2)}{2} = 2.
\end{aligned}$$

Example. Show that the limit does not exist: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+2y}{x-2y}$

Solution.

If the limits do not agree for any two paths, then it is not continuous at that point. So, if we take a path along the x-axis, we get

$$\lim_{(x,0)\to(0,0)} \frac{x+2(0)}{x-2(0)} = \lim_{(x,0)\to(0,0)} \frac{x}{x} = \lim_{(x,0)\to(0,0)} 1 = 1.$$

and if we take a path along the y-axis, we get

$$\lim_{(0,y)\to(0,0)} \frac{(0)+2y}{(0)-2y} = \lim_{(x,0)\to(0,0)} \frac{2y}{-2y} = \lim_{(x,0)\to(0,0)} -1 = -1.$$

The two limits do not agree, so the limit does not exist.

Assignment

Worksheet 06:

https://mathpost.asu.edu/~wells/math/teaching/mat272_spring2015/homework06.pdf

As always, you may work in groups, but every member must individually submit a homework assignment.