Recitation 05: Curvature and Normal Vectors; Planes and Surfaces

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Suppose $\mathbf{r}$ is a smooth parametrized curve. If s denotes arc length and $\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}$ is the unit tangent vector, the *curvature* is $\kappa(s) = \left|\frac{d\mathbf{T}}{ds}\right|$.

It's important to note that κ is parametrized with respect to arc length. If our curve is *not* given an arc length parametrization, it would be nice to be able to give an explicit formula for the curvature of the parametrized curve in terms of its native parametrization. And so we have the following result:

Theorem 1. *Let* $\mathbf{r}$ *be a smooth curve parametrized by* t*,* $\mathbf{v} = \mathbf{r}'$ *the velocity vector, and* $\mathbf{T}$ *the unit tangent vector. Then the curvature is given by*

$$\kappa(s) = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{|\mathbf{T}'(\mathbf{t})|}{|\mathbf{r}'(\mathbf{t})|}.$$

Alternatively,

Theorem 2. *Let* $\mathbf{r}$ *be a smooth curve (with any parametrization),* $\mathbf{v} = \mathbf{r}'$ *the velocity vector, and* $\mathbf{a} = \mathbf{v}' = \mathbf{r}''$ *the acceleration vector. Then the curvature is given by*

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}.$$

Given a smooth parametrized curve $\mathbf{r}$, the *(principal) unit normal vector* is

$$\mathbf{N}(s) = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$

In practice, we use

$$\mathbf{N}(t) = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt}.$$

It's important to note that $\mathbf{T}(s) \perp \mathbf{N}(s)$ for every s. Because each is a unit vector and they are orthogonal, these form (part of) an *orthonormal basis* for vectors on the curve.

Theorem 3. *For the acceleration* $\mathbf{a}$*, we have that we can write it as*

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T},$$

where the normal component $a_N = \kappa|\mathbf{v}|^2 = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|}$ *and the tangential component* $a_T = \frac{d^2s}{dt^2} = \frac{\mathbf{v}\cdot\mathbf{a}}{|v|}$.

Example. Find $\mathbf{T}(1)$, $\mathbf{N}(t)$, and $\mathbf{N}(1)$ for the curve $\mathbf{r}(t) = \langle 3t, 2t^2 \rangle$.

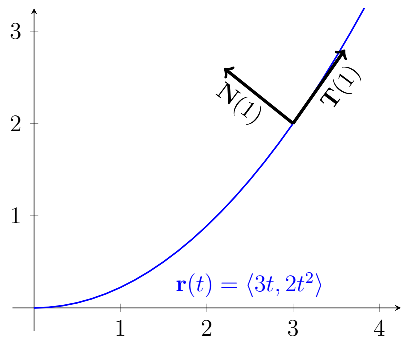
Solution. By differentiating, we get $\mathbf{r}'(t) = \langle 3, 4t \rangle$ and $|\mathbf{r}'(t)| = \sqrt{9 + 16t^2}$. Thus

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{9+16t^2}} \langle 3, 4t \rangle \quad \Rightarrow \quad \mathbf{T}(1) = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle.$$

By differentiating again (and with much simplification), we also have that $\mathbf{T}'(t) = \frac{12}{(9+16t^2)^{3/2}} \langle -4t, 3 \rangle$ and $|\mathbf{T}'(t)| = \frac{12}{9+16t^2}$. Thus

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{1}{\sqrt{9+16t^2}} \langle -4t, 3 \rangle \quad \Rightarrow \quad \mathbf{N}(1) = \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle.$$

The graph of the previous example is below. As you can see, the normal and tangent vectors are orthogonal to each other, as claimed



Example. Find the tangential and normal components of acceleration for $\mathbf{r}(t) = \langle 3t, -t, t^2 \rangle$.

By differentiating, we get

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 3, -1, 2t \rangle,$$
$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = \langle 0, 0, 2 \rangle.$$

So, by the above theorems, the tangential component is given by

$$a_T = \frac{\mathbf{v} \cdot \mathbf{a}}{|v|} = \frac{4t}{\sqrt{10 + 4t^2}}$$

and the normal component is given by

$$a_N = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|} = \frac{|\langle -2, -6, 0 \rangle|}{\sqrt{10 + 4t^2}} = \frac{2\sqrt{10}}{\sqrt{10 + 4t^2}}.$$

This material on this slide is not necessary for the class, but it does help to complete the greater picture and explain why we look to find a basis relative to points on the curve.

The plane spanned by these two vectors $\mathbf{T}$ and $\mathbf{N}$ is called the *osculating plane*

To complete the 3-dimensional orthonormal basis for vectors along our curve, we define the *unit binormal vector* $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. The 3-dimensional space spanned by $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ is called the *TNB frame* or *Frenet frame*. The *torsion* (which measures how fast the curve is pulling away from the osculating plane) is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \frac{\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{|\mathbf{v} \times \mathbf{a}|^2}.$$

Given a fixed point P and a nonzero vector $\mathbf{n}$, the plane in $\mathbb{R}^3$ is the set of points in $Q \in \mathbb{R}^3$ for which $\overrightarrow{PQ}$ is orthogonal to $\mathbf{n}$. In particular, if $\mathbf{n} = \langle a, b, c\rangle$ and $P = (x_0, y_0, z_0)$, the plane can be represented by the equation $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

Two planes are *parallel* if their normal vectors are parallel; they are *orthogonal* if their normal vectors are orthogonal.

Assignment

Worksheet 05:

`https://mathpost.asu.edu/~wells/math/teaching/mat272_spring2015/homework05.pdf`

As always, you may work in groups, but every member must individually submit a homework assignment.