Recitation 04: Motion in Space; Length of Curves

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We may often want to think about vector-valued functions as modeling motion in space. Suppose the function $\mathbf{r}(t) = \langle r_1(t), \ldots, r_n(t) \rangle$ represents a particle's *position* in space at time t. Then, just as you probably learned in Calc I or Physics, $\mathbf{v}(t) = \mathbf{r}'(t)$ is the particle's *velocity* and $|\mathbf{v}(t)|$ is the particle's *speed.* Lastly, $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$ is the particle's *acceleration.*

Example. Given the following position function, determine the velocity, speed, and acceleration of the object. $\mathbf{r}(t) = \langle 8\sin(t), 8\cos(t)\rangle$, $0 \leq t \leq 2\pi$.

Solution.

Knowing how to derive the acceleration function from the position or velocity functions is. From physics, we know that the force vector **F** of an object is directly proportional to the mass m of that object and its acceleration vector **a**, i.e.

$$\mathbf{F} = m\mathbf{a}(t) = m\mathbf{v}'(t) = m\mathbf{r}''(t).$$

Example. Find the force acting on an object of mass 100 kg with the given position function (units of meters and seconds): $\mathbf{r}(t) = \langle \cos(4t), \sin(5t) \rangle$.

Solution.

With a few derivatives, we get that

$$\mathbf{F} = m\mathbf{a}(t) = m\mathbf{r}'' = 100\langle -16\cos(4t), -25\sin(5t) \rangle.$$

For two-dimensional motion and an object acted upon only by gravity (say, throwing a ball into the air at angle θ from the ground), with an initial position $\langle x_0, y_0 \rangle$ and initial velocity $\mathbf{v}_0 = \langle |\mathbf{v}_0| \cos(\theta), |\mathbf{v}_0| \sin(\theta) \rangle$, we can deduce the following:

$$\begin{aligned}
\text{time of flight} = T &= \frac{2|\mathbf{v}_0| \sin(\theta)}{g} \\
\text{range} &= \frac{|\mathbf{v}_0|^2 \sin(2\theta)}{g} \\
\text{maximum height} &= y\left(\frac{T}{2}\right) = \frac{(|\mathbf{v}_0| \sin(\theta))^2}{2g}.
\end{aligned}$$

Just as you did back in Calc II, we can determine the length of a curve traversing through n-dimensional space. Let $\mathbf{r}(t) = \langle r_1(t), \ldots, r_n(t) \rangle$ be such a curve and suppose $r_1', \ldots, r_n'$ are continuous. Then the arc length of the interval $[a, b]$ is given by

$$\begin{aligned} L &= \int_a^b |\mathbf{r}'(t)|\, dt \\ &= \int_a^b \sqrt{[r_1'(t)]^2 + \cdots + [r_n'(t)]^2}\, dt. \end{aligned}$$

If given a polar curve $r = f(\theta)$, then the arc length of the interval $[\alpha, \beta]$ is given by

$$L = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta.$$

Example. Find the length of the following curve: $\mathbf{r}(t) = \left\langle \frac{5}{2}\sin(2t), \frac{13}{2}\cos(2t), \frac{12}{2}\sin(2t) \right\rangle$, $0 \leq t \leq \pi$.

Solution.

Using the arc length formula, we have

$$\begin{aligned} L &= \int_0^\pi \sqrt{[5\cos(2t)]^2 + [-13\sin(2t)]^2 + [12\cos(2t)]^2}\, dt \\ &= \int_0^\pi \sqrt{169\cos^2(2t) + 169\sin^2(2t)}\, dt \\ &= 13 \int_0^\pi dt \\ &= 13\pi. \end{aligned}$$

Assignment

Worksheet 04:

`https://mathpost.asu.edu/~wells/math/teaching/mat272_spring2015/homework04.pdf`

As always, you may work in groups, but every member must individually submit a homework assignment.