Recitation 10: Geometric Series

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Recall

Definition. A *series* is a limit of sums of terms b_k, and is given by $\sum_{k=0}^{\infty} b_k = \lim_{n\to\infty} \sum_{k=0}^{n} b_k$

Definition. A *geometric series* is a series of the form $\sum_{k=0}^{\infty} ar^k$ (where r is called the ratio).

If the upper limit of the summation is finite, we have that $\sum_{k=0}^{n} ar^k = \frac{1-r^n}{1-r}$. If we take the limit as $n \to \infty$, we get $\sum_{k=0}^{\infty} ar^k = \lim_{n\to\infty} \sum_{k=0}^{n} ar^k$. What conditions do we need to determine whether or not this limit converges? $|r| < 1$.

Definition. A *telescoping series* is a series that has only finitely many terms after cancellation.

Example (Book §8.3, #7). Evaluate the following geometric sum: $\sum_{k=0}^{8} 3^k$

$$\sum_{k=0}^{8} 3^k = 1 \cdot \frac{1-3^9}{1-3} = \frac{19682}{2} = 9841$$

Example (Book §8.3, #11). Evaluate the following geometric sum: $\sum_{k=0}^{9} \left(-\frac{3}{4}\right)^k$

$$\sum_{k=0}^{9} \left(-\frac{3}{4}\right)^k = 1 \cdot \frac{1-\left(-\frac{3}{4}\right)^{10}}{1+\frac{3}{4}} = \frac{4^{10}-3^{10}}{4^{10}-3\cdot 4^9} = \frac{141,361}{262,144} \approx 0.54$$

Example (Book §8.3, #19)**.** Evaluate the geometric series, or state that it diverges: $\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k$

$\left|\frac{1}{4}\right| < 1$, so it converges to $\displaystyle\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$.

Example (Book §8.3, #29)**.** Evaluate the geometric series, or state that it diverges: $\sum_{k=4}^{\infty}\frac{1}{5^k}$

$\left|\frac{1}{5}\right| < 1$, so it converges to $\displaystyle\sum_{k=4}^{\infty}\frac{1}{5^k} = \frac{\frac{1}{5^4}}{1-\frac{1}{5}} = \frac{1}{5^4-5^3} = \frac{1}{500}$.

Example (Book §8.3, #47)**.** Find a formula for the n-th term of the sequence of partial sums $\{S_n\}$. Then evaluate $\lim_{n\to\infty} S_n$ to obtain the value of the series or state that the series diverges: $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)$

Notice that

$$
\begin{aligned}
S_1 &= \frac{1}{2}-\frac{1}{3}\\
S_2 &= \frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}=\frac{1}{2}-\frac{1}{4}\\
&\vdots\\
S_n &= \frac{1}{2}-\frac{1}{n+2}=\frac{n}{2n+4}.
\end{aligned}
$$

Then $\displaystyle\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{n}{2n+4}=\frac{1}{2}.$

Example (Book §8.3, #51)**.** Find a formula for the n-th term of the sequence of partial sums $\{S_n\}$. Then evaluate $\lim_{n\to\infty} S_n$ to obtain the value of the series or state that the series diverges: $\displaystyle\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right)$

Notice that, $\ln\left(\frac{k+1}{k}\right) = \ln(k+1) - \ln(k)$, so

$$\begin{aligned}
S_1 &= \ln(2) - \ln(1) \\
S_2 &= \ln(2) - \ln(1) + \ln(3) - \ln(2) = \ln(3) - \ln(1) \\
&\vdots \\
S_n &= \ln(n+1) - \ln(1) = \ln(n+1).
\end{aligned}$$

Then $\displaystyle\lim_{n\to\infty} S_n = \lim_{n\to\infty} \ln(n+1)$, which diverges.

Example (Book §8.3, #53)**.** Find a formula for the n-th term of the sequence of partial sums $\{S_n\}$. Then evaluate $\lim_{n\to\infty} S_n$ to obtain the value of the series or state that the series diverges: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k+p)(k+p+1)}$, where p is a positive integer.

By partial fractions, $\frac{1}{(k+p)(k+p+1)} = \frac{1}{k+p} - \frac{1}{k+p+1}$, so

$$
\begin{aligned}
S_1 &= \frac{1}{1+p} - \frac{1}{2+p} \\
S_2 &= \frac{1}{1+p} - \frac{1}{2+p} + \frac{1}{2+p} - \frac{1}{3+p} = \frac{1}{1+p} - \frac{1}{3+p} \\
&\vdots \\
S_n &= \frac{1}{1+p} - \frac{1}{n+p+1} = \frac{n}{(1+p)(n+p+1)} = \frac{n}{n(p+1)+(p+1)^2}.
\end{aligned}
$$

Then $\displaystyle\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{n(p+1)+(p+1)^2} = \frac{1}{p+1}$.

Assignment

Recitation Notebook:
§8.3 - #1, #2, #3, #4, #5

As always, you may work in groups, but every member must individually submit a homework assignment.