Recitation 10: Series Convergence Tests

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Example (Book §8.3, #53)**.** Find a formula for the n-th term of the sequence of partial sums $\{S_n\}$. Then evaluate $\lim_{n\to\infty} S_n$ to obtain the value of the series or state that the series diverges: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k+p)(k+p+1)}$, where p is a positive integer.

By partial fractions, $\frac{1}{(k+p)(k+p+1)} = \frac{1}{k+p} - \frac{1}{k+p+1}$, so

$$\begin{aligned}
S_1 &= \frac{1}{1+p} - \frac{1}{2+p} \\
S_2 &= \frac{1}{1+p} - \frac{1}{2+p} + \frac{1}{2+p} - \frac{1}{3+p} = \frac{1}{1+p} - \frac{1}{3+p} \\
&\vdots \\
S_n &= \frac{1}{1+p} - \frac{1}{n+p+1} = \frac{n}{(1+p)(n+p+1)} = \frac{n}{n(p+1)+(p+1)^2}.
\end{aligned}$$

Then $\displaystyle\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{n(p+1)+(p+1)^2} = \frac{1}{p+1}.$

Theorem (Divergence Test). *If* $\sum a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$*, and if* $\lim_{n\to\infty} a_n \neq 0$*, then* $\sum a_n$ *diverges.*

Theorem (Integral Test). *If* f *is positive, continuous, and decreasing for* $x \geq 1$ *and* $a_n = f(n)$*, then*

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\,dx$$

either both converge or diverge.

Theorem (p-Series Test)**.** *The p-series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

- *converges if* $p > 1$, *and*
- *diverges if* $0 < p \leq 1$.

Theorem (Ratio Test)**.** *Let* $\sum a_n$ *be a series with nonzero terms.*

- $\sum a_n$ *converges absolutely if* $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.
- $\sum a_n$ *diverges if* $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ *or* $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$.
- *The Ratio Test is inconclusive if* $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$.

Theorem (Root Test). *Let $\sum a_n$ be a series.*

- *$\sum a_n$ converges absolutely if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1$.*
- *$\sum a_n$ diverges if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \infty$.*
- *The Root Test is inconclusive if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$.*

Theorem (Comparison Test). *Let $0 \leq a_n \leq b_n$ for all n.*

- *If $\sum\limits_{n=1}^{\infty} b_n$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ converges.*
- *If $\sum\limits_{n=1}^{\infty} a_n$ diverges, then $\sum\limits_{n=1}^{\infty} b_n$ diverges.*

Example. Determine whether the series converges or diverges. $\sum_{n=1}^{\infty} \frac{n}{\ln(n)}$

$$\lim_{n\to\infty} \frac{n}{\ln(n)} \overset{L'H}{=} \lim_{n\to\infty} \frac{1}{1/x} = \infty,$$

so the series diverges by the Divergence Test.

Example. Determine whether the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Since $f(x) = \frac{1}{x^2+1}$ satisfies the conditions for the integral test

$$\int_1^{\infty} \frac{dx}{x^2+1} = \lim_{r\to\infty} \int_1^r \frac{dx}{x^2+1} = \lim_{r\to\infty} \arctan(x)\Big|_1^r = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4},$$

so the series converges by the Integral Test.

Example. Determine whether the series converges or diverges. $\sum_{n=1}^{\infty} 2n^{-3/2}$

$$\sum_{n=1}^{\infty} 2n^{-3/2} = 2\sum_{n=1}^{\infty} \frac{1}{n^{3/2}},$$

so the series diverges by the p-Series Test.

Example. Determine whether the series converges or diverges. $\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2 2^{n+2} 3^{-(n+1)}}{n 2^{n+1} 3^{-n}} \right| = \lim_{n\to\infty} \frac{2(n+1)^2}{3n^2} = \frac{2}{3} < 1$$

so the series converges by the Ratio Test

Example. Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{e^{2n}}{n^n}} = \lim_{n\to\infty} \frac{e^2}{n} = 0 < 1$$

so the series converges by the Root Test

Example. Determine whether the series converges or diverges $\sum_{k=1}^{\infty} \frac{1}{2+\sqrt{n}}$.

The series looks similar to a divergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$, but alas $\frac{1}{2+\sqrt{n}} \leq \frac{1}{\sqrt{n}}$ for each $n > 1$. So compare it to the harmonic series. Then $a_n = \frac{1}{n} \leq \frac{1}{2+\sqrt{n}} = b_n$, so since

$\sum a_n$ diverges, so does $\sum b_n$ by the Comparison Tests

Assignment

Recitation Notebook:
§8.4 - #1, #2, #3, #4, #5, #6

As always, you may work in groups, but every member must individually submit a homework assignment.