Recitation 04: Improper Integrals

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Example. Evaluate the following integral, or state that it diverges $\displaystyle\int_2^{\infty} \frac{x}{(x+2)^2}\,dx$.

Solution.

We'll use the substitution $u = x + 2$ annd $du = dx$. Then

$$\begin{aligned}
\int_2^{\infty} \frac{x}{(x+2)^2}\,dx &= \lim_{n\to\infty} \int_{x=2}^{x=n} \frac{x}{(x+2)^2}\,dx \\
&= \lim_{n\to\infty} \int_{x=2}^{x=n} \frac{u-2}{u^2}\,du \\
&= \lim_{n\to\infty} \int_{x=2}^{x=n} \left[\frac{1}{u} - \frac{2}{u^2}\right] du \\
&= \lim_{n\to\infty} \left[\ln(u) + \frac{2}{u}\right]_{x=2}^{x=n} \\
&= \lim_{n\to\infty} \left[\ln(x+2) + \frac{2}{x+2}\right]_{x=2}^{x=n} \\
&= \lim_{n\to\infty} \left[\ln(t+2) + \frac{2}{t+2} - \ln(4) - \frac{1}{2}\right]
\end{aligned}$$

$$= -\infty + 0 - \ln(4) - \frac{1}{2} = -\infty$$

Thus the integral diverges.

Example. Use symmetry to evaluate the following integral $\int_{-\infty}^{\infty} e^{-|t|}\, dt$.

Solution.

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-|t|}\, dt &= 2\int_{0}^{\infty} e^{-t}\, dt \\
&= \lim_{n\to\infty} 2\int_{0}^{n} e^{-t}\, dt \\
&= \lim_{n\to\infty} -2e^{-t}\Big|_{0}^{n} \\
&= \lim_{n\to\infty} 2 - 2e^{-n} \\
&= 2.
\end{aligned}$$

Example. Determine if the following integral is convergent or divergent. If it converges, find its value. $\displaystyle\int_0^3 \frac{dx}{\sqrt{3-x}}$

Solution.

Since the integrand is undefined at $x = 3$,

$$\begin{aligned}
\int_0^3 \frac{dx}{\sqrt{3-x}} &= \lim_{n\to 3} \int_0^n \frac{dx}{\sqrt{3-x}} \\
&= \lim_{n\to 3} -2\sqrt{3-x}\,\Big|_0^n \\
&= \lim_{n\to 3} -2\sqrt{3-n} + 2\sqrt{3} \\
&= 2\sqrt{3}.
\end{aligned}$$

Example. Determine if the following integral is convergent or divergent. If it converges, find its value. $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$

Solution.

The important part of this integral is setting it up. Since both of our limits of integration are infinity, we'll need to split the integral up into two separate integrals, and then take the limits.

$$\begin{aligned}
\int_{\infty}^{\infty} xe^{-x^2}\,dx &= \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_{0}^{\infty} xe^{-x^2}\,dx \\
&= \lim_{m\to-\infty} \int_{m}^{0} xe^{-x^2}\,dx + \lim_{n\to\infty} \int_{0}^{n} xe^{-x^2}\,dx
\end{aligned}$$

Assignment

Recitation Notebook:
§7.7 - #1, #2, #3, #4, #5

As always, you may work in groups, but every member must individually submit a homework assignment.