Recitation 02: Integration by Parts & Trig Integrals

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Recall the theorems:

Theorem (Integration by Parts)**.** *Suppose that u and v are differentiable functions. Then*

$$\int u\,dv = uv - \int v\,du$$

Theorem (Integration by Parts - Definite Integral)**.** *Suppose that u and v are differentiable on (a, b). Then*

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,dx$$

Example (Integration by Parts)**.** Evaluate the following integral:

$$\int \theta \sec^2(\theta)\, d\theta$$

Solution.

Let $u = \theta$ and $dv = \sec^2(\theta)\, d\theta$. Then $du = d\theta$ and $v = \tan(\theta) + C$. So

$$\begin{aligned}\int \theta \sec^2(\theta)\, d\theta &= \theta \tan(\theta) - \int \tan(\theta)\, d\theta \\ &= \theta \tan(\theta) - \ln|\sec(\theta)| + C\end{aligned}$$

Example (Repeated Integration by Parts)**.** Evaluate the following integral:

$$\int t^3 e^{-t}\, dt$$

Solution.

Since t^3 is a polynomial, we should probably pick $u = t^3$ and $dv = e^{-t}, dt$. However, doing so will require us to do integration by parts 3 times. Polynomials are actually kind of special in that multiple derivatives will eventually get us to zero. As a result, there is a faster way through it (the Tabular Method).

We form a table with two columns. The left column is u and the right column is dv. In the left column, we take a derivative of each preceding row until we get to 0. In the right column, we take an anti-derivative of each preceding row. Our table thus looks like this:

u	dv
t^3	e^{-t}
$3t^2$	$-e^{-t}$
$6t$	e^{-t}
6	$-e^{-t}$
0	e^{-t}

From here, we alternately assign $\pm$ to the u terms and associate each u to the dv in the row below.

	u		dv
$+$	t^3	$\searrow$	e^{-t}
$-$	$3t^2$	$\searrow$	$-e^{-t}$
$+$	$6t$	$\searrow$	e^{-t}
$-$	6	$\searrow$	$-e^{-t}$
	0		e^{-t}

So then we multiply associated u and dv terms together and add them to get

$$\int t^3 e^{-t}\, dt = -t^3 e^{-t} - 3t^2 e^{-t} - 6te^{-t} - 6e^{-t} + C.$$

Example (Definite Integrals (Integration by Parts))**.** Evaluate the following definite integral:

$$\int_0^{\pi/2} x\cos(2x)\,dx$$

Solution.

Let $u = x$ and $dv = \cos(2x)\,dx$. Then $du = dx$ and $v = \frac{1}{2}\sin(2x) + C$, so

$$\begin{aligned}
\int_0^{\pi/2} x\cos(2x)\,dx &= \frac{1}{2}x\sin(2x)\Big|_0^{\pi/2} - \frac{1}{2}\int_0^{\pi/2} \sin(2x)\,dx \\
&= \frac{1}{2}x\sin(2x)\Big|_0^{\pi/2} + \frac{1}{4}\cos(2x)\Big|_0^{\pi/2} \\
&= (0-0) + (-\frac{1}{4} - \frac{1}{4}) = -\frac{1}{2}.
\end{aligned}$$

Example (Integrals of $\sin(x)$ and $\cos(x)$)**.** Evaluate the following integral:

$$\int \sin^2(x)\cos^2(x)\,dx$$

Solution.

Since both sin and cos have even powers, we use the half-angle identities to rewrite the integrand.

Recall that $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$ and $\sin^2(x) = \frac{1}{2}(1-\cos(2x)$. So

$$\begin{aligned}\int \sin^2(x)\cos^2(x) &= \frac{1}{4}\int (1-\cos(2x))\,(1+\cos(2x))\;dx\\ &= \frac{1}{4}\int \left[1-\cos^2(2x)\right]\;dx\\ &= \frac{1}{4}\int \left[1-\frac{1}{2}\,(1+\cos(4x))\right]\;dx\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\int dx - \frac{1}{8}\int dx - \frac{1}{8}\int \cos(4x)\,dx \\
&= \frac{1}{4}x - \frac{1}{8}x - \frac{1}{32}\sin(4x) = \frac{x}{8} - \frac{\sin(4x)}{32}.
\end{aligned}
$$

Example (Integrals of $\tan(x)$ and $\sec(x)$)**.** Evaluate the following integral:

$$\int \sec^2(x)\tan^{1/2}(x)\,dx$$

Solution.

Since sec has an even power, we choose $u = \tan(x)$ and change any leftover $\sec^{2k}(x)$ terms into $(\tan^2(x)+1)^k$. So, with our choice of u, we get that $du = \sec^2(x)\,dx$, and thus

$$\begin{aligned}\int \sec^2(x)\tan^{1/2}(x)\,dx &= \int u^{1/2}\,du \\ &= \frac{2}{3}u^{3/2} + C \\ &= \frac{2}{3}\tan^{3/2}(x) + C.\end{aligned}$$

Example (Square roots)**.** Evaluate the following integral:

$$\int_0^{\pi/2} \sqrt{1-\cos(2x)}\, dx$$

Solution.

Again, we appeal to the half-angle formula $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$ and get

$$\begin{aligned}\int_0^{\pi/2} \sqrt{1-\cos(2x)}\, dx &= \sqrt{2}\int_0^{\pi/2} \sin(x)\, dx \\ &= -\sqrt{2}\cos(x)\Big|_0^{\pi/2} \\ &= \sqrt{2}.\end{aligned}$$

Assignment

Recitation Notebook:
§7.1 - #1, #2, #6
§7.2 - #1, #3, #4

As always, you may work in groups, but homework must written up and submitted by each individual.