Recitation 14: Integration with Substitution

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Example (§5.5, #1). If the change of variables $u = x^2 - 4$ is used to evaluate the definite integral $\int_2^4 f(x)\,dx$, what are the new limits of integration?

Solution.

The idea here is that adjusting our variables requires us to change our limits of integration. We want to move from the limits "$x = 2$ to $x = 4$" to "$u =$? to $u =$??." Since we're given the substitution, it simply becomes

$$\int_{x=2}^{x=4} f(x)\,dx = \int_{u=(2)^2-4}^{u=(4)^2-4} g(u)\,du = \int_{u=0}^{u=12} g(u)\,du.$$

Example (§5.5, #2)**.** Use a change of variables to find the following indefinite integral. Check your work by differentiation. $\displaystyle\int \frac{(\sqrt{x}+1)^4}{2\sqrt{x}}\,dx$

Solution.

We want to substitute for the less pretty terms, so it seems like $u = \sqrt{x}$ or $u = \sqrt{x}+1$ would make reasonable candidates. Since the latter will potentially reduce our integral even more, let's try it

$$u = \sqrt{x}+1 \qquad\qquad du = \frac{1}{2\sqrt{x}}\,dx$$

Beautiful. We can then make the substitution and integrate.

$$\int \frac{(\sqrt{x}+1)^4}{2\sqrt{x}}\,dx = \int u^4\,du = \frac{1}{5}u^5 + C = \frac{1}{5}(\sqrt{x}+1)^5 + C.$$

I'll leave the derivative check to you.

Example (§5.5, #3). Use a change of variables to find the following indefinite integral. Check your work by differentiation. $\displaystyle\int \sin^{10}(\theta)\cos(\theta)\,d\theta$

Solution.

Looking ahead at what our potential du will be, let

$$u = \sin(\theta) \qquad\qquad du = \cos(\theta)\,d\theta.$$

Then, substituting and integrating,

$$\int \sin^{10}(\theta)\cos(\theta)\,d\theta = \int u^{10}\,du = \frac{1}{11}u^{11} + C = \frac{1}{11}\sin^{11}(\theta) + C.$$

Once again, I'll leave the differentiation up to you.

Example (§5.5, #4)**.** Use a change of variables to find the following definite integral.
$$\int_0^4 \frac{p}{\sqrt{9+p^2}}\,dp$$

Solution.

Letting

$$u = \sqrt{9+p^2} \qquad\qquad du = \frac{p}{\sqrt{9+p^2}}\,dp,$$

our integral becomes nigh trivial:

$$\int_{p=0}^{p=4} \frac{p}{\sqrt{0+p^2}}\,dp = \int_{u=3}^{u=5} du = u\Big|_3^5 = 2$$

Example (§5.5, #5)**.** Use a change of variables to find the following definite integral.
$$\int_0^6 \frac{dx}{x^2+36}$$

Solution.

This one is a little less obvious, but we're shooting for our integral to be an arctan function. In order to do that, we need want to factor out a 36 in the denominator, so let

$$u = \frac{1}{6}x \qquad\qquad du = \frac{1}{6}\,dx.$$

Then we can evaluate

$$\begin{aligned}\int_{x=0}^{x=6} \frac{1}{x^2+36}\,dx &= \int_{u=0}^{u=1} \frac{6}{(6u)^2+36}\,du \\ &= \int_{u=0}^{u=1} \frac{6}{36u^2+36}\,du\end{aligned}$$

$$= \frac{1}{6} \int_{u=0}^{u=1} \frac{1}{u^2 + 1} \, du$$

$$= \frac{1}{6} \arctan(u) \Big|_{u=0}^{u=1}$$

$$= \frac{\pi}{24}.$$

Example (§5.5, #6). Evaluate the following integrals in which the function f is unspecified. Note $f^(p)$ is the pth derivative of f and f^p is the pth power of f. Assume f and its derivatives are continuous for all real numbers. $\int (5f^3(x) + 7f^2(x) + f(x))f'(x)\,dx$.

Solution.

Making the substitution

$$u = f(x) \qquad du = f'(x)\,dx,$$

we get that

$$\begin{aligned}\int (5f^3(x) + 7f^2(x) + f(x))f'(x)\,dx &= \int 5u^3 + 7u^2 + u\,du \\ &= \frac{5}{4}u^4 + \frac{7}{3}u^3 + \frac{1}{2}u^2 + C\end{aligned}$$

$$= \frac{5}{4} f^4(x) + \frac{7}{3} f^3(x) + \frac{1}{2} f^2(x) + C.$$

Assignment

None.