Recitation 12

Joseph Wells
Arizona State University

May 3, 2014

Example (§4.7, #6)**.** Evaluate the following limit or explain why it does not exist. Check your results by graphing.

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{\ln x}$$

solution

This equation limit is currently in a "1^∞" indeterminate form, so we have to do a bit of conversion using the fact that $f(x)^{g(x)} = e^{\ln f(x)^{g(x)}} = e^{g(x)\ln f(x)}$. So, we can deduce that

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{\ln x} = e^L$$

where

$$L = \lim_{x\to\infty}\ln(x)\ln\left(1+\frac{1}{x}\right) \qquad (\infty\cdot 0 \text{ form})\,.$$

Now, we want to use L'Hopital's on this function, but it's still not in one of the correct indeterminate forms, so we rewrite it as

$$L = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{\ln(x)}} \qquad (0/0 \text{ form})\,.$$

Applying L'Hopital's, we get

$$\begin{aligned}
L &\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}\ln\left(1+\frac{1}{x}\right)}{\frac{d}{dx}\frac{1}{\ln(x)}} \\
&= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x\ln(x)^2}} \\
&= \lim_{x\to\infty} \frac{\ln(x)^2}{x^2+x} \qquad (\infty/\infty \text{ form}) \\
&\overset{L'H}{=} \lim_{x\to\infty} -\frac{\frac{d}{dx}\ln(x)^2}{\frac{d}{dx}x^2+x} \\
&= \lim_{x\to\infty} \frac{\frac{2\ln(x)}{x}}{2x+1}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{x\to\infty} \frac{2\ln(x)}{2x^2+x} \qquad\qquad (\infty/\infty \text{ form}) \\
&\overset{L'H}{=} \lim_{x\to\infty} -\frac{\frac{d}{dx}2\ln(x)}{\frac{d}{dx}2x^2+x} \\
&= \lim_{x\to\infty} \frac{\frac{2}{x}}{4x+1} \\
&= \lim_{x\to\infty} \frac{2}{4x^2+x} \\
&= 0.
\end{aligned}
$$

So plugging back into one of our original equations, we get

$$
\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{\ln(x)} = e^L = e^0 = 1.
$$

Example (§4.8, #4)**.** Consider the following descriptions of the vertical motion of an object subject only to the acceleration due to gravity.

A stone is thrown vertically upward with a velocity of 30 m/s from the edge of a cliff 200 m above a river.

1. Find the velocity of the object for all relevant times.
2. Find the position of the object for all relevant times.
3. Find the time when the object reaches its highest point? (What is the height?)
4. Find the time when the object strikes the water.

solution

The force of gravity is $g = -9.81\text{m/s}^2$. For simplicity in calculation, however, we'll just estimate $g = -10\text{m}/s^2$. We begin with the equation representing the object's acceleration at a given time:

$$a(t) = g = -10$$

1. If you recall a previous lecture, we briefly discussed that the acceleration was the derivative of the velocity, so we can deduce the equation of the object's velocity:

$$\begin{aligned} a(t) = v'(t) &= g \\ v(t) &= gt + v_0 \\ &= -10t + 30 \qquad (t \geq 0) \end{aligned}$$

2. Similarly, the velocity is the derivative of the position, so

$$\begin{aligned} v(t) = y'(t) &= gt + v_0 \\ y(t) &= \frac{g}{2}t^2 + v_0 t + y_0 \\ &= -5t^2 + 30t + 200 \quad (t \geq 0) \end{aligned}$$

3. The object reaches its highest point when the derivative of the position function is zero (ie, when $v(t) = 0$), so

$$\begin{aligned} v(t) &= 0 = -10t + 30 \\ &\Rightarrow t = 3\text{s}. \end{aligned}$$

At 3 seconds, the height of the object (above the water) is

$$y(3) = -5(9) + 30(3) + 200 = 245\text{m}.$$

4. The object reaches the water when $y(t) = 0$, so

$$\begin{aligned} y(t) = 0 &= -5t^2 + 30t + 200 \\ &= -5(t^2 - 6t - 40) \\ &= -5(t - 10)(t + 4) \\ \Rightarrow t &= 10\text{s}. \end{aligned}$$

Assignment

MAT270 Recitation Notebook
§4.7, Problems 2,4,5
§4.8, Problems 1,2