Recitation 09: Critical Points & First Derivative Test

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Example (§4.1, #3)**.** Find the critical points of f. Assume a and b are constants.

$$f(x) = x^3 - 3ax^2 + 3a^2x - a^3 + b$$

Solution.

As usual, we will take the derivative, set it equal to 0, and solve.

$$\begin{aligned} 0 &= f'(x) \\ &= 3x^2 - 6ax + 3a^2 \\ &= 3(x-a)^2, \end{aligned}$$

so critical points occur when $x = a$.

Example (§4.1, #4)**.**

$$f(x) = x^{2/3}(4 - x^2); \quad [-3, 4]$$

a. Find the critical points of the following functions on the given interval.

b. Use a graphing device to determine whether the critical points correspond to *local maxima*, *local minima*, or *neither*.

c. Find the *absolute maximum* and *absolute minimum* values on the given interval.

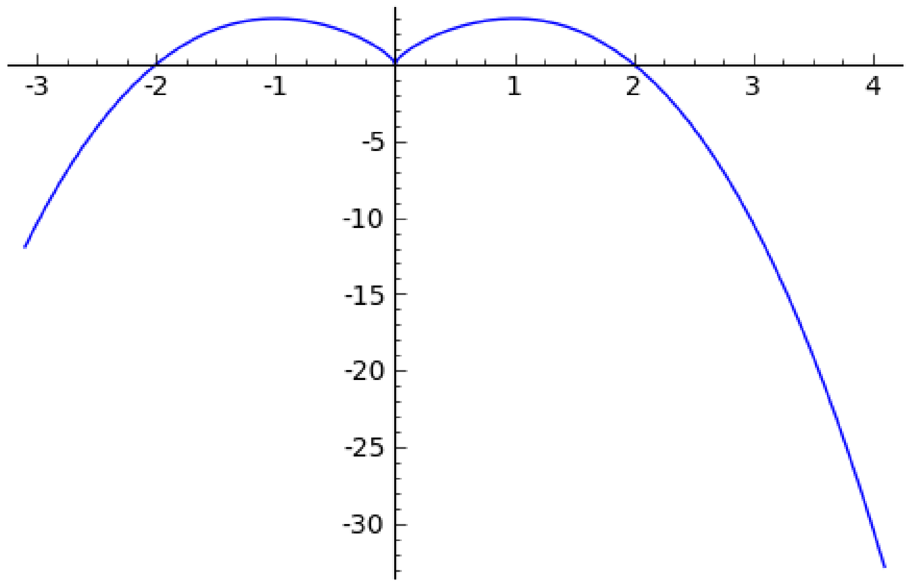
Solution.

a. Take the derivative, set it equal to 0, and solve.

$$\begin{aligned} 0 &= f'(x) \\ &= \frac{d}{dx}\left[x^{2/3}(4-x^2)\right] \\ &= \frac{d}{dx}\left[4x^{2/3}-x^{8/3}\right] \\ &= \frac{8}{3}x^{-1/3}-\frac{8}{3}x^{5/3} \\ &= \frac{8}{3}x^{-1/3}(1-x^2) \\ &= \frac{8}{3}x^{-1/3}(1-x)(1+x). \end{aligned}$$

Notice that only $x = \pm 1$ make the equation zero. $x^{-1/3} = \frac{1}{x^{1/3}}$ is undefined at $x = 0$, but our definition of a critical point also includes places where the first derivative is undefined, so critical points occur when $x = -1, 0, 1$.

b.



It's clear from the picture that $x = \pm 1$ correspond to local maxima, $x = 0$ corresponds to a local minimum, and $x = 3, 4$ are neither (why? local extrema only occur on the interior of the interval we're examining).

c. To calculate we absolute extrema, we calculate the values of f at all of our critical points and at the endpoints.

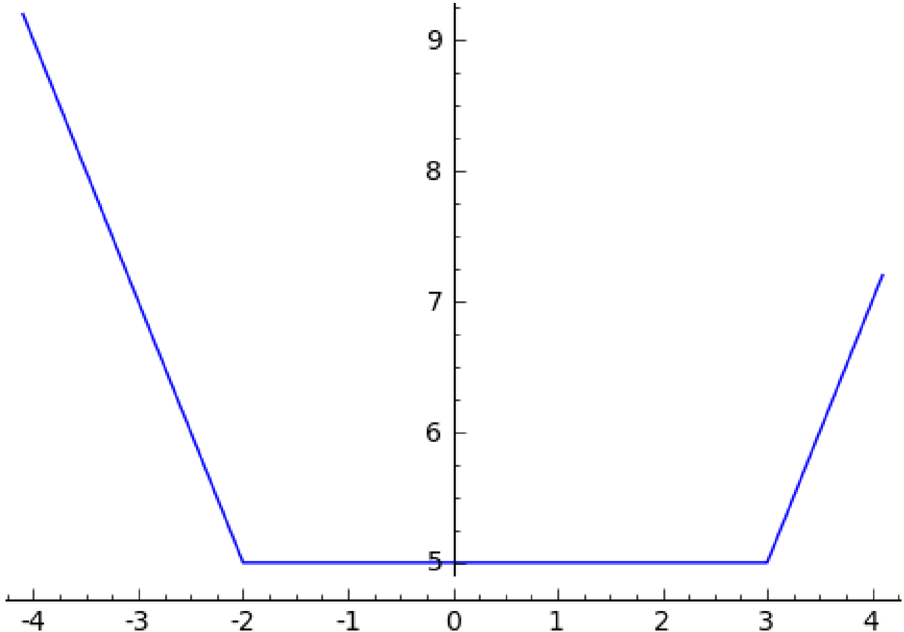
$$\begin{aligned} f(-3) &\approx -10.4 \\ f(-1) &= 3 \\ f(0) &= 0 \\ f(1) &= 3 \\ f(4) &\approx -30.2 \end{aligned}$$

It's clear from this table that $f(-1) = f(1) = 3$ is our absolute maximum and that $f(4) \approx -30.2$ is our absolute minimum of the interval.

Example (§4.1, #5)**.** Graph the following function and determine the local and absolute extreme values on the given interval.

$$f(x) = |x - 3| + |x + 2|; \quad [-4, 4]$$

Solution.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From the graph, we can see that $x = 4$ corresponds to our absolute maximum and the whole interval $I = [-2, 3]$. Since I is on the interior of our interval, I corresponds to our local minima. There are no local maxima.

Example (§4.2, #5)**.** Determine whether the following statements are true and give an explanation or a counterexample.

a If $f'(x) > 0$ and $f''(x) < 0$ on an interval, then f is increasing at a decreasing rate.

b If $f'(c) > 0$ and $f''(c) = 0$, then f has a local maximum at c.

c Two functions that differ by a constant increase and decrease on the same intervals.

d If f and g increase on an interval, then the product fg also increases on that interval.

e There exists a function f that is continuous $(-\infty, \infty)$ with exactly three critical points, all of which correspond to local maxima.

Solution.

a True. The first derivative tells us about the rate of change of the original function. The second derivative tells us the rate of increase/decrease.

b False. Consider $f(x) = x$ and $c = 2$. Then $f'(2) = 1 > 0$ and $f''(2) = 0$, but the function has no local extrema.

c True. Presumably, "differ by a constant" refers to addition of a constant, which is ultimately zero when we take the first derivative.

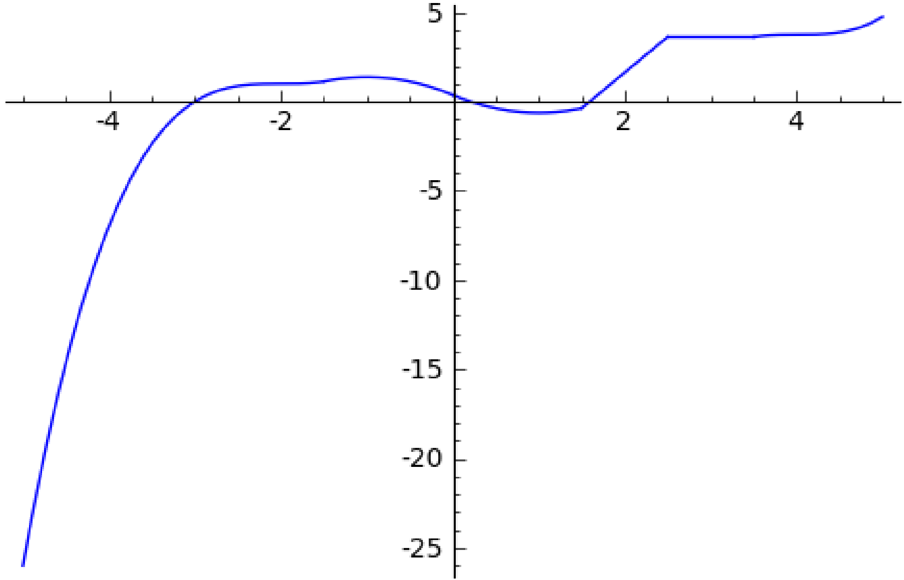
d False. Via the product rule, we see that $(fg)' = f'g + g'f$. Since $f(x)$ and $g(x)$ are not necessarily positive numbers, it's entirely possible that $(fg)'$ can be less than or equal to zero, so not increasing.

e False. Suppose there did exist some function $f(x)$ and x-values $a < b < c$ such that $f'(a) = f'(b) = f'(c) = 0$ and further suppose that $f(b)$ is a local maximum. Then we have for all $x \in (a, b)$ that $f'(x) > 0$. If a were also a local maximum, then for all $x \in (a, b)$, we have that $f'(x) < 0$, which contradicts a necessary requirement for $f(b)$ to be a local maximum.

Example (§4.2, #6)**.** Sketch the graph of a function that is continuous on $(-\infty, \infty)$ and satisfies the following set of conditions:

$$\begin{aligned} f''(x) &> 0 \quad \text{on } (-\infty, -2); \\ f''(-2) &= 0; \\ f'(-1) = f'(1) &= 0; \\ f''(2) &= 0; \\ f'(3) &= 0; \\ f''(x) &> 0; \quad \text{on } (4, \infty); \end{aligned}$$

Solution.



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Assignment

MAT270 Recitation Notebook
§4.1, Problems 1,2
§4.2, Problems 1,2,4