Recitation 07: Implicit and Logarithmic Differentiation

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May 3, 2014

Up until this point, we've taken derivatives of functions that can be solved explicitly for y in terms of x; in notation, we write $y = f(x)$. For example,

$$y = 3x + 5$$
$$y' = \frac{dy}{dx} = \frac{d}{dx}[3x + 5] = 3$$

However, such functions might not always be expressible in terms of one variable, and we might have to consider functions in terms of both x and y; notationally, $F(x, y)$ (I use the capital F to make the difference clear, though later in the course we'll reserve capital letter functions for antiderivatives). For example, consider

$$y^2 = x$$

If we want to find $y' = \frac{dy}{dx}$, we would first have to split it into two functions: $y = +\sqrt{x}$ and $y = -\sqrt{x}$, and then take the derivative of each. *Implicit differentiation* allows us to find y' without first splitting apart the equation and solving for y in terms of x (which may not always be possible). To do this, we let $y = f(x)$ and take our derivatives as usual

$$\begin{aligned}
y^2 &= x \\
(f(x))^2 &= x \\
\frac{d}{dx}\left[(f(x))^2\right] &= \frac{d}{dx}[x] \\
2f(x)f'(x) &= 1 \\
f'(x) &= \frac{1}{2f(x)} \\
y' &= \frac{1}{2y}
\end{aligned}$$

Example (§3.7, #2). Use implicit differentiation to find $\frac{dy}{dx} = y'$ for the equation $(xy+1)^3 = x - y^2 + 8$.

$$\begin{aligned}
\frac{d}{dx}\left[(xy+1)^3\right] &= \frac{d}{dx}\left[x - y^2 + 8\right] \\
3(xy+1)^2\frac{d}{dx}[xy+1] &= 1 - 2yy' \\
3(xy+1)^2(y + xy') &= 1 - 2yy' \\
3y(xy+1)^2 y + 3xy'(xy+1)^2 &= 1 - 2yy' \\
2yy' + 3xy'(xy+1)^2 &= 1 - 3x(xy+1)^2 \\
y'(2y + 3x(xy+1)^2) &= 1 - 3x(xy+1)^2 \\
y' &= \frac{1 - 3x(xy+1)^2}{2y + 3x(xy+1)^2}
\end{aligned}$$

Logarithms, and in particular $\ln = \log_e$, play a large role in calculus. Quickly, we review some of their properties:

$$
\begin{aligned}
\ln\left(ab\right) &= \ln(a) + \ln(b) \\
\ln\left(\frac{a}{b}\right) &= \ln(a) - \ln(b) \\
\ln\left(a^b\right) &= b\ln(a) \\
\log_a\left(b\right) &= \frac{\ln(b)}{\ln(a)}
\end{aligned}
$$

Although I wont go into the proof here, it can be shown that

$$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}$$

Knowing this, we can derive each of the following:

$$\frac{d}{dx}\left[\log_a(x)\right] = \frac{1}{x\ln(a)}$$
$$\frac{d}{dx}\left[a^x\right] = a^x\ln(a)$$

The logarithm gives rise to the technique of *logarithmic differentiation*, which can be particularly useful in situations where the complexity of the product, chain, and quotient rules would make even the hardest of mathematicians cry. To see how it works, consider an arbitrary function $f(x)$. Using the chain rule, we can solve for $f'(x)$:

$$\frac{d}{dx}\left[\ln\left(f(x)\right)\right] = \frac{f'(x)}{f(x)}$$

so we can rearrange this to get that

$$f'(x) = f(x)\frac{d}{dx}\left[\ln\left(f(x)\right)\right]$$

Example (§3.8, #4). Use logarithmic differentiation to evaluate $f'(x)$ for $f(x) = \dfrac{x^8 \cos^3(x)}{\sqrt{x-1}}$

We'll start by looking at $\ln(f(x))$:

$$\begin{aligned}
\ln(f(x)) &= \ln\left(\frac{x^8 \cos^3(x)}{\sqrt{x-1}}\right) \\
&= \ln(x^8) + \ln(\cos^3(x)) - \ln(\sqrt{x-1}) \\
&= 8\ln(x) + 3\ln(\cos(x)) - \frac{1}{2}\ln(x-1),
\end{aligned}$$

so

$$\begin{aligned}
\frac{d}{dx}[\ln(f(x))] &= 8\frac{d}{dx}[\ln(x)] + 3\frac{d}{dx}[\ln(\cos(x))] - \frac{1}{2}\frac{d}{dx}[\ln(x-1)] \\
&= \frac{8}{x} - \frac{3\sin(x)}{\cos(x)} - \frac{1}{2(x-1)}.
\end{aligned}$$

Putting it all together, we get

$$f'(x) = f(x)\frac{d}{dx}[\ln(f(x))] = \frac{x^8\cos^3(x)}{\sqrt{x-1}}\left(\frac{8}{x} - \frac{3\sin(x)}{\cos(x)} - \frac{1}{2(x-1)}\right).$$

Assignment

MAT270 Recitation Notebook
§3.7, Problems 1,3,4
§3.8, Problems 2,3