Recitation 04: Formal Limits and Derivatives

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This week you were given the formal definition of a limit:

Definition. Suppose $f(x)$ exists for all x in some open interval containing a except possibly at a. Then

$$\lim_{x \to a} f(x) = L$$

if for any *varepsilon* > 0, there exists some $\delta > 0$ such that

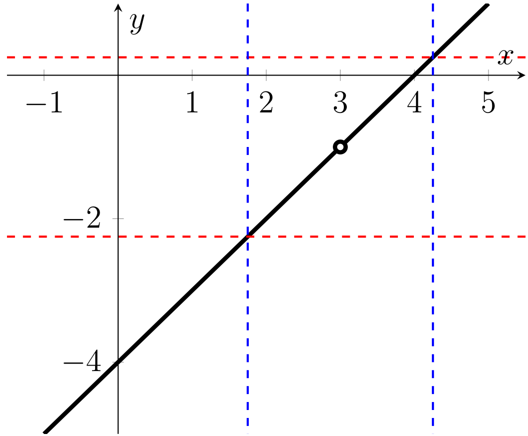
$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

What does this mean? In so many, words, it means that for any restriction on the range of our function around L, we can find a sufficient restriction on the domain of our function around a such that for any $x \in (a - \delta, a + \delta)$, $f(x) \in (L - \varepsilon, L + \varepsilon)$.

It might help to consider the following concrete example.

Example. Prove that $\displaystyle\lim_{x \to 3} \frac{x^2 - 7x + 12}{x - 3} = -1$

Before we start the proof, we draw a graph of the function to try to illustrate the concept.



Given some $\varepsilon > 0$, we want to show that we can find a $\delta > 0$ such that, for any x in the interval $(3 - \delta, 3 + \delta)$, we have that $f(x)$ is in the interval $(-1 - \varepsilon, -1 + \varepsilon)$.

Proof. Let $\varepsilon > 0$ be an arbitrary real number. Choose $\delta = \varepsilon$ (this choice will become apparent later) and suppose that $0 < |x - 3| < \delta$. We then have that

$$\begin{aligned} |f(x) - L| &= \left| \frac{x^2 - 7x + 12}{x - 3} - (-1) \right| \\ &= \left| \frac{(x-3)(x-4)}{x - 3} + 1 \right| \\ &= |(x - 4) + 1| \\ &= |x - 3| < \delta = \varepsilon. \end{aligned}$$

Therefore, by definition, -1 is the limit of $f(x)$ as x approaches 3. $\square$

Now we move on to a very important part of calculus: the derivative.

Back in section 2.1, we discussed secant lines and made some conjectures about the slope of a tangent line when examining the slopes of secant lines as the secant line's interval became smaller and smaller. Now that we have limits under our belt, we are able to formalize the notion of the tangent line.

Definition. The slope of the tangent line is

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

and the equation of the tangent line is

$$y - f(a) = m(x - a)$$

In fact, this tangent line at a point is just a specific instance of the derivative, which is a function in terms of x. The formal definition of the derivative is

Definition. Provided the limit exists, the derivative of f is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

If $f'(x)$ exists, we say that f is differentiable at x. If f is differentiable at every point of an open interval (a, b), we say that f is differentiable on (a, b).

Example. Do MAT270 Recitation Notebook, §3.1, Problem 6. Create the graph of a continuous function $y = f(x)$, where

$$f'(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } 0 < x < 1, \\ -1 & \text{if } x > 1. \end{cases}$$

Assignment

MAT270 Recitation Notebook
§2.7, Problems 2,4
§3.1, Problems 1,3,4,6