Recitation 03: Limits

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Recall that removable discontinuities are those holes in the domain of the function that can be ignored when calculating the limit at that point.

For example, consider $f(x) = \dfrac{x^2+x-2}{x+1}$. The domain of $f(x)$ is $(-\infty, 1) \cup (1, \infty)$, but we can still calculate the limit at $x = 1$:

$$\lim_{x\to 1} \frac{x^2+x-2}{x+1} = \lim_{x\to 1} \frac{(x-1)(x+2)}{x-1} = \lim_{x\to 1} x+2 = 3.$$

Another type of discontinuity is the vertical asymptote, where the limit goes to positive or negative infinity as x approaches some real number a. We can determine these analytically by first removing the removable discontinuities and analyzing the leftover function.

Consider $f(x) = \dfrac{x^2+x-2}{x^2-1}$. Then

$$\lim_{x\to -1^+} \frac{x^2+x-2}{x^2-1} = \lim_{x\to -1^+} \frac{(x-1)(x+2)}{(x-1)(x+1)} = \lim_{x\to -1^+} \frac{x+2}{x+1} = \infty$$

since the numerator approaches 1 from the right and the denominator approaches 0 from the right.

There are also horizontal asymptotes that bound the range of the function as x tends toward $\pm\infty$. These require a bit more trickery and application of our limit laws (Theorem 2.3 in the book). I've included two examples.

Example 1. Let $f(x) = 13 + \dfrac{7}{x+6}$ and find $\lim_{x\to\infty} f(x)$. Well, we have

$$\lim_{x\to\infty} 13 + \frac{7}{x+6} = \left(\lim_{x\to\infty} 13\right) + \left(\lim_{x\to\infty} \frac{7}{x+6}\right) = 13 + 0 = 13.$$

Example 2. Let $f(x) = 2x + 1 - \sqrt{4x^2+5}$ and find $\lim_{x\to\infty} f(x)$. It's not obvious in this form what our limit would be, so we need to cleverly change the form of the function.

$$\begin{aligned} f(x) &= 2x + 1 - \sqrt{4x^2 + 5} \\ &= \frac{2x + 1 - \sqrt{4x^2 + 5}}{1} \cdot \frac{2x + 1 + \sqrt{4x^2 + 5}}{2x + 1 + \sqrt{4x^2 + 5}} \\ &= \frac{4x - 4}{2x + 1 + \sqrt{4x^2 + 5}}. \end{aligned}$$

What we notice is that, for sufficiently large x, $\lim_{x\to\infty} \sqrt{4x^2 + 5} = \lim_{x\to\infty} 2x$, so

$$\lim_{x\to\infty} \frac{4x - 4}{2x + 1 + \sqrt{4x^2 + 5}} = \lim_{x\to\infty} \frac{4x - 4}{2x + 1 + 2x} = \lim_{x\to\infty} \frac{4x - 4}{4x + 1}.$$

Now we notice that $\lim_{x\to\infty} \sqrt{4x - 4} = \lim_{x\to\infty} 4x$, and similarly for the denominator, so

$$\lim_{x\to\infty} \frac{4x - 4}{4x + 1} = \lim_{x\to\infty} \frac{4x}{4x} = \lim_{x\to\infty} 1 = 1.$$

Recall that a function $f(x)$ is **continuous at** $x = a$ if $\lim_{x\to a} f(x) = f(a)$, that is, if the limit exists and agrees with the function value. We say that the function is **continuous** if it is continuous at every point in its domain.

We also have a sweet theorem for continuity that is more-or-less analogous to Theorem 2.3 for limits:

Theorem (2.9 - Continuity Rules)**.** *If f and g are continuous at a, then the following functions are continuous at a. Assume c is a constant.*

a. $f \pm g$

b. cf

c. fg

d. $\frac{f}{g}$, *if* $g(a) \neq 0$

Theorem (2.15 - Intermediate Value Theorem)**.** *Suppose f is continuous on the interval $[a, b]$ and L is a number between $f(a)$ and $f(b)$. Then there is at least one number c in (a, b) satisfying $f(c) = L$.*

Assignment

MAT270 Recitation Notebook
§2.4, Problems 3
§2.5, Problems 1,4
§2.6, Problems 4,6