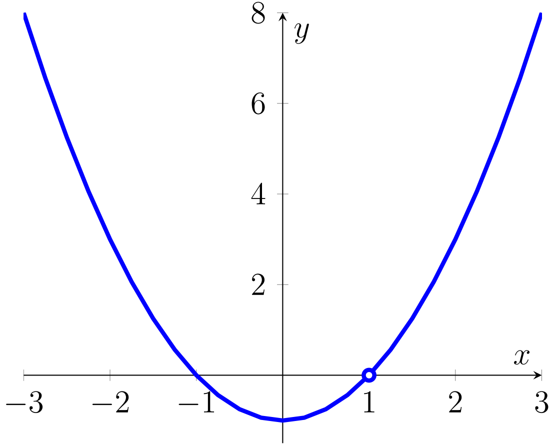
Recitation 02: Limits

Joseph Wells
Arizona State University

May 3, 2014

Limits tell us how a function behaves as it gets close to a point, even if it's not defined at that point. We can think of limits approaching some value from the left, as well as limits from the right. When do we say that the limit exists? If the two are equal and both do not approach infinity. Consider the function $f(x) = \frac{x^3 - x^2 - x + 1}{x-1}$:



The function is not defined at $x = 1$, but

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$$

so the limit exists. In particular, we can determine the limit analytically by factoring the numerator in $f(x)$:

$$\lim_{x \to 1} \frac{x^3 - x^2 - x + 1}{x - 1} = \lim_{x \to 1} \frac{(x^2 - 1)(x - 1)}{x - 1} = \lim_{x \to 1} x^2 - 1 = 0.$$

Let $f(x) = \frac{x^3 - x^2 - x + 1}{x - 1}$ and $g(x) = x^2 - 1$. Is it true that $f(x) = g(x)$? Consider the domain of each function. We see that the domain of $f(x)$ is $(-\infty, 1) \cup (1, \infty)$ and the domain of $g(x)$ is $(-\infty, \infty)$, so in fact, $f(x) \neq g(x)$.

We also have a couple of theorems that can make calculating limits easier on us. The proofs are discussed in your book, so for brevity, we'll state them as fact.

Theorem (2.3 - Limit Laws). *Suppose* $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *exist. Let* $c \in \mathbb{R}$ *be any real number and* $m, n > 0$ *be integers.*

1. ***Sum*** $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. ***Difference*** $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. ***Constant Multiples*** $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$
4. ***Product*** $\lim_{x\to a} [f(x)g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right]$
5. ***Quotients*** $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, *provided* $\lim_{x\to a} g(x) \neq 0$
6. ***Powers*** $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$
7. ***Fractional Powers*** $\lim_{x\to a} [f(x)]^{n/m} = \left[\lim_{x\to a} f(x)\right]^{n/m}$, *provided* $f(x) \geq 0$ *for* x *near* a *if* m *is even and* n/m *is reduced to lowest terms.*

Theorem (2.5 - Squeeze Theorem). *Suppose the functions f, g, h satisfy $f(x) \leq g(x) \leq h(x)$ for all x near a (except possibly at a). If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.*

Consider the function $f(x) = \frac{\sin(x)}{x}$, the "sinc function". We want to determine $\lim_{x\to\infty} f(x)$. What is the range of $\sin(x)$? $\sin(x)$ has a range of $[-1, 1]$, so it must be that

$$-\frac{1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}$$

Since

$$\lim_{x\to\infty} -\frac{1}{x} = \lim_{x\to\infty} \frac{1}{x} = 0,$$

we can apply the Squeeze theorem to get that $\lim_{x\to\infty} \dfrac{\sin(x)}{x} = 0$.

Assignment

MAT270 Recitation Notebook
§2.2, Problems 3,4
§2.3, Problems 1,2,3
§2.4, Problems 1

I also recommend looking at #4 and #6 from section §2.3. Neither will be due, but you should know how to handle fractilons with radicals and limits with function composition.