MAT266 Homework 04 (Solutions)

1. We'll begin by making the substitution $x = -Mv^2/(2RT)$, $dx = -Mv/(RT)\,dv$. Notice that as $v \to \infty$, $x \to -\infty$. So our integral becomes

$$\begin{aligned}
\overline{v} &= \frac{4}{\sqrt{\pi}}\left(\frac{M}{2RT}\right)^{3/2}\int_0^\infty v^3 e^{-Mv^2/(2RT)}\,dv \\
&= \frac{4}{\sqrt{\pi}}\left(\frac{M}{2RT}\right)^{3/2}\left(-\frac{RT}{M}\right)\left(-\frac{2RT}{M}\right)\int_0^{-\infty} xe^x\,dx \\
&= \sqrt{\frac{8RT}{\pi M}}\int_0^{-\infty} xe^x\,dx \\
&= \lim_{t\to-\infty}\sqrt{\frac{8RT}{\pi M}}\int_0^t xe^x\,dx \\
&= \lim_{t\to-\infty}\sqrt{\frac{8RT}{\pi M}}\left[xe^x - e^x\right]_0^t && \text{(integration by parts)} \\
&= \lim_{t\to-\infty}\sqrt{\frac{8RT}{\pi M}}\left[\left(te^t - e^t\right) - (0-1)\right] \\
&= \sqrt{\frac{8RT}{\pi M}}.
\end{aligned}$$

Note that evaluating the limit did require the use of L'Hospital's Rule as we have an "$\infty \cdot 0$" indeterminate form.

2. We note that, because k is negative, as $t \to \infty$, $e^k \to 0$ (this will be useful in evaluating the limit).

$$\begin{aligned}
M &= -k\int_0^\infty te^{kt}\,dt \\
&= \lim_{b\to\infty} -k\int_0^b te^{kt}\,dt \\
&= \lim_{b\to\infty} -k\left[\frac{1}{k}te^{kt} - \frac{1}{k^2}e^{kt}\right]_0^b && \text{(integration by parts)} \\
&= \lim_{b\to\infty} -k\left[\left(\frac{1}{k}be^{kb} - \frac{1}{k^2}e^{kb}\right) - \left(0 - \frac{1}{k^2}\right)\right] \\
&= \lim_{b\to\infty}\left[-be^{kb} + \frac{1}{k}e^{kb} - \frac{1}{k}\right] && \text{(“}\infty\cdot 0\text{” indeterminate form)} \\
&= \lim_{b\to\infty}\left[-\frac{b}{e^{-kb}} + \frac{1}{k}e^{kb} - \frac{1}{k}\right] && \text{(“}\infty/\infty\text{” indeterminate form)} \\
&= \lim_{b\to\infty}\left[-\frac{1}{-ke^{-kb}} + \frac{1}{k}e^{kb} - \frac{1}{k}\right] && \text{(L'Hospital's rule)} \\
&= \left[0 + 0 - \frac{1}{k}\right] = -\frac{1}{0.000121} \approx 8264.4628
\end{aligned}$$

3. Using Simpson's Rule with $\Delta x = 2\,\mathrm{m}$ we have that the swimming pool area is approximately

$$\frac{2\,\mathrm{m}}{3}\left[0\,\mathrm{m} + 4(6.2\,\mathrm{m}) + 2(7.2\,\mathrm{m}) + 4(6.8\,\mathrm{m}) + 2(5.6\,\mathrm{m}) + 4(5.0\,\mathrm{m}) + 2(4.8\,\mathrm{m}) + 4(4.8\,\mathrm{m}) + 0\,\mathrm{m}\right] \approx 84.267\,\mathrm{m}^2.$$

4. Recall that the annual profits with continual growth rate r are given by $A(t) = \$893,000e^{rt}$ (where t is in years). The total profit is found by integrating $A(t)$, and so the difference in cumulative total profit over 5 between at 3.5% growth rate and a 5% growth rate is

$$\$893,000\int_0^5 e^{0.05t} - e^{0.035t}\,dt = \$893,000\left[\frac{1}{0.05}e^{0.05t} - \frac{1}{0.035}e^{0.035t}\right]_0^5 = \$193,183$$

5. Using the Trapezoid Rule with $\Delta x = 1.5\,\mathrm{cm}$ we have that the liver's volume is approximately

$$\frac{1.5\,\mathrm{cm}}{2}\left[0\,\mathrm{cm}^2 + 2(18\,\mathrm{cm}^2) + 2(58\,\mathrm{cm}^2) + 2(79\,\mathrm{cm}^2) + 2(94\,\mathrm{cm}^2) + 2(106\,\mathrm{cm}^2) + 2(117\,\mathrm{cm}^2) + \right.$$
$$\left. +2(128\,\mathrm{cm}^2) + 2(63\,\mathrm{cm}^2) + 2(39\,\mathrm{cm}^2) + 0\,\mathrm{cm}^2\right] = 1053\,\mathrm{cm}^3$$

6. We can view this as a solid of revolution in the following way: Place the center of the base circle at the origin. The slanted edge of the frustum passes through the points $(R, 0)$ and (r, h), and so the line representing this edge is precisely $y = \left(\frac{h}{r-R}\right)(x - R)$. Since we'll be using the disk method, we integrate with respect to y. At each fixed y, the radius of this disk is $x = \left(\frac{r-R}{h}\right)y + R$, and so the volume of the solid is

$$\begin{aligned} V &= \int_0^h \pi x^2\,dy \\ &= \pi\int_0^h \left(\frac{r-R}{h}y + R\right)^2\,dy \\ &= \pi\int_0^h \left(\frac{r-R}{h}\right)^2 y^2 + 2\pi\frac{R(r-R)}{h}y + \pi R^2\,dy \\ &= \pi\left[\frac{1}{3}\left(\frac{r-R}{h}\right)^2 y^3 + \frac{R(r-R)}{h}y^2 + R^2 y\right]_0^h \\ &= \boxed{\frac{1}{3}\pi h\left(r^2 + rR + R^2\right)} \end{aligned}$$