MAT266 Homework 03 (Solutions)

1. Making the substitution

$$u = e^m$$
$$du = e^m\,dm \quad \Rightarrow \quad dm = \frac{1}{u}\,du$$

we get

$$\int \sqrt{e^{2m}+1}\,dm = \int \frac{\sqrt{u^2+1}}{u}\,du = \sqrt{e^{2m}+1} - \ln\left|\frac{1+\sqrt{e^{2m}+1}}{e^m}\right| + C.$$

2. We begin with integration by parts, taking

$$u = \arcsin(\sqrt{x}) \qquad du = \frac{1}{2\sqrt{x}\sqrt{1-x}}\,dx$$
$$dv = dx \qquad v = x$$

The integration by parts formula then yields

$$\int \arcsin(\sqrt{x})\,dx = x\arcsin(\sqrt{x}) - \frac{1}{2}\int \frac{\sqrt{x}}{\sqrt{1-x}}\,dx$$

Now making the substitution

$$w = \sqrt{x}$$
$$dw = \frac{1}{2\sqrt{x}}\,dx \quad \Rightarrow \quad 2w\,dw = dx$$

we get

$$x\arcsin(\sqrt{x}) - \frac{1}{2}\int \frac{1}{\sqrt{x}\sqrt{1-x}}\,dx = x\arcsin(\sqrt{x}) - \int \frac{w^2}{\sqrt{1-w^2}}\,dx$$

and applying the given formula results in

$$\begin{aligned} x\arcsin(\sqrt{x}) - \int \frac{w^2}{\sqrt{1-w^2}}\,dx &= x\arcsin(\sqrt{x}) + \frac{w}{2}\sqrt{1-w^2} - \frac{1}{2}\arcsin(w) + C \\ &= x\arcsin(\sqrt{x}) + \frac{\sqrt{x}}{2}\sqrt{1-x} - \frac{1}{2}\arcsin(\sqrt{x}) + C \end{aligned}$$

3. From the table we see that $\Delta x = 0.3$, and so

$$\begin{aligned} \int_0^{1.5} g(x)\,dx &\approx \frac{\Delta x}{2}\left[g(0.3) + 2g(0.6) + 2g(0.9) + 2g(1.2) + g(1.5)\right] \\ &= 0.15\left[8 + 2(7.7) + 2(6) + 2(6.2) + 2(5.9) + 4.1\right] = 9.555 \end{aligned}$$

4. With $n = 10$ intervals, we have that $\Delta x = \frac{(10^{-6}+10^{-6})}{10} = 2 \times 10^{-7}$. Each interval's left endpoint x_j (for $j = 0, \ldots, 9$) is going to be given by

$$x_j = -10^{-6} + 2j \times 10^{-7} = -10 \times 10^{-7} + 2j \times 10^{-7} = (-10 + 2j) \times 10^{-7}$$

which means the midpoints of $[x_j, x_{j+1}]$ (for $j = 0, \ldots, 9$) are given by

$$\begin{aligned}\frac{1}{2}(x_j + x_{j+1}) &= \frac{1}{2}\left[(-10+2j) \times 10^{-7} + (-10 + 2(j+1)) \times 10^{-7}\right] \\ &= \frac{1}{2}\left[(-18+4j) \times 10^{-7}\right] = (-9+2j) \times 10^{-7}.\end{aligned}$$

With $N = 10000$, $d = 10^{-4}$, $\lambda = 632.8 \times 10^{-9}$, we have that $I(\theta)$ is given by

$$I(\theta) = \frac{10^8 \sin^2 k}{k^2}, \quad \text{where} \quad k = \frac{\pi \sin\theta}{632.8 \times 10^{-9}},$$

and thus our midpoint approximation is given by

$$\begin{aligned}\int_{-10^{-6}}^{10^{-6}} I(\theta)\, d\theta &\approx \Delta x \left[I(-9 \times 10^{-7}) + I(-7 \times 10^{-7}) + I(-5 \times 10^{-7}) + \cdots + I(9 \times 10^{-7})\right] \\ &\approx 59.4103\,\text{cd}.\end{aligned}$$

5. From the table we see that $\Delta t = 30\,\text{min} = 0.5\,\text{hr}$, and so

$$\begin{aligned}E = \int_0^6 P(t)\, dt &\approx S_{12} \\ &= \frac{\Delta t}{3}\left[P(0) + 4P(0.5) + 2P(1) + 4P(1.5) + 2P(2) + \cdots + 4P(5.5) + P(6)\right] \\ &\approx 10177.33\,\text{MWh}\end{aligned}$$