MAT266 Calculus for Engineering II

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Contents

5 Integrals **3**

5.5 The Substitution Rule . . . 3

6 Techniques of Integration **10**

6.1 Integration by Parts . . . 10

6.2 Trigonometric Integrals and Substitutions . . . 16

6.2.1 Trigonometric Integrals . . . 16

6.2.2 Trigonometric Substitutions . . . 20

6.3 Partial Fractions . . . 23

6.3.1 Using a Matrix to Solve Systems of Linear Equations . . . 28

6.4 Integration with Tables and Computer Algebra Systems . . . 29

6.5 Approximate Integration . . . 31

6.6 Improper Integrals . . . 36

6.6.1 Type I: Infinite Interval . . . 36

6.6.2 Type II: Discontinuous Integrand . . . 38

6.6.3 A Comparison Test for Improper Integrals . . . 40

7 Applications of Integration **41**

7.1 Area Between Curves . . . 41

7.2 Volumes . . . 44

7.3 Volumes by Cylindrical Shells . . . 50

7.4 Arc Length . . . 56

7.6 Applications to Physics and Engineering . . . 60

8 Series **64**

8.1 Sequences . . . 64

8.2 Series . . . 67

8.3 The Integral and Comparison Tests . . . 72

8.4 Other Convergence Tests . . . 74

8.4.1 Alternating Series . . . 74

8.4.2 Ratio Test . . . 75

8.5 Power Series . . . 77

8.6 Representing Functions as Power Series . . . 80

8.7 Taylor and Maclaurin Series . . . 83

8.7.1 Taylor Polynomials . . . 87

9 Parametric Equations and Polar Coordinates **89**

9.1 Parametric Curves . . . 89

9.2 Calculus with Parametric Curves . . . 92

9.2.1 Slopes and Tangent Lines . . . 92

9.2.2 Arc Length . . . 94

9.2.3 Areas . . . 96

9.3 Polar Coordinates . . . 98

9.3.1 Tangents to Polar Curves . . . 100

9.4 Areas and Lengths in Polar Coordinates . . . 102

9.4.1 Area in Polar Coordinates . . . 102

9.4.2 Arc Length in Polar Coordinates . . . 104

9.4.3 A fun application of polar coordinates . . . 105

5 Integrals

5.5 The Substitution Rule

Example 5.5.1 (Warm Up)**.** Find $\frac{d}{dx}\left[(3x^2-5)^8\right]$.

Using the chain rule, we have

$$\frac{d}{dx}\left[(3x^2-5)^8\right] = 8\left(3x^2-5\right)^7 \cdot 6x.$$

Recall that the chain rule says

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x),$$

where $g(x)$ is your "inner function" and $f(x)$ is your "outer function". Recall also that, if $u = g(x)$ is a differentiable function, then in the language of differentials, we have $du = g'(x)\,dx$.

The following rule combines these two concepts in a way that is exactly analogous to the chain rule for differentiation.

Proposition 5.5.2 (Substitution Rule)**.** *If $u = g(x)$ is a differentiable function whose range is an interval I, and if f is continuous on I, then*

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du$$

Example 5.5.3. Using the substitution rule, evaluate $\int 8\left(3x^2-5\right)^7 \cdot 6x\,dx$.

To apply the substitution rule, we first find $g(x)$, our "inner function".

$$u = g(x) = 3x^2 - 5.$$
$$du = g'(x)\,dx = 6x\,dx.$$

Hence, by the substitution rule,

$$\begin{aligned}\int 8\left(3x^2-5\right)^7 \cdot 6x\,dx &= \int 8u^7\,du \\ &= u^8 + C \\ &= \left(3x^2-5\right)^8 + C \qquad \text{(substitute back in for } u\text{)}.\end{aligned}$$

Since the derivative of this function is exactly the original integrand, we indeed have the correct answer.

Example 5.5.4. Using the substitution rule, evaluate $\int e^{-2x}\,dx$.

To apply the substitution rule, we first find $g(x)$, our "inner function".

$$\begin{aligned} u &= g(x) = -2x \\ du &= g'(x)\,dx = -2\,dx \quad \Rightarrow \quad -\frac{1}{2}\,du = dx. \end{aligned}$$

Hence, by the substitution rule,

$$\begin{aligned} \int e^{-2x}\,dx &= \int e^{u}\left(-\frac{1}{2}\right)\,du \\ &= -\frac{1}{2}\int e^{u}\,du \\ &= -\frac{1}{2}e^{u} + C \\ &= -\frac{1}{2}e^{-2x} + C \qquad \text{(substitute back in for } u\text{).} \end{aligned}$$

Since the derivative of this function is exactly the original integrand, we indeed have the correct answer.

Example 5.5.5. Using the substitution rule, evaluate $\int \frac{(\ln x)^2}{x}\,dx$.

To apply the substitution rule, we first find $g(x)$, our "inner function".

$$\begin{aligned} u &= g(x) = \ln x \\ du &= g'(x)\,dx = \frac{1}{x}\,dx. \end{aligned}$$

Hence, by the substitution rule,

$$\begin{aligned} \int \frac{(\ln x)^2}{x}\,dx &= \int u^2\,du \\ &= \frac{1}{3}u^3 + C \\ &= \frac{1}{3}(\ln x)^3 + C \qquad \text{(substitute back in for } u\text{).} \end{aligned}$$

Since the derivative of this function is exactly the original integrand, we indeed have the correct answer.

Example 5.5.6. Using the substitution rule, evaluate $\displaystyle\int \frac{x^3}{\sqrt{x^2+1}}\,dx$.

To apply the substitution rule, we first find $g(x)$, our "inner function".

$$\begin{aligned} u &= g(x) = x^2 + 1 \qquad (5.5.1)\\ du &= g'(x)\,dx = 2x\,dx \quad \Rightarrow \quad \frac{1}{2}\,du = x\,dx \end{aligned}$$

This gives us

$$\int \frac{x^3}{\sqrt{x^2+1}}\,dx = \int \frac{x^2}{\sqrt{u}}\left(\frac{1}{2}\right)\,du$$

But what do we do with the x^2 term? Well notice that we can rearrange Equation 5.5.1 to get $x^2 = u - 1$, so

$$\begin{aligned} &= \frac{1}{2}\int \frac{u-1}{\sqrt{u}}\,du \\ &= \frac{1}{2}\int \left(u^{1/2} - u^{-1/2}\right)\,du \\ &= \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right) + C \\ &= \frac{1}{3}u^{3/2} - u^{1/2} + C \\ &= \frac{1}{3}\left(x^2+1\right)^{3/2} - \left(x^2+1\right)^{1/2} + C \qquad \text{(substitute back in for } u\text{).} \end{aligned}$$

Since the derivative of this function is exactly the original integrand, we indeed have the correct answer.

Example 5.5.7. Using the substitution rule, evaluate $\displaystyle\int \tan(x)\,dx$.

To apply the substitution rule, we first find $g(x)$, our "inner function". But where can this come from? First we recall that $\tan x = \dfrac{\sin x}{\cos x}$ and let

$$\begin{aligned} u &= g(x) = \cos x \\ du &= g'(x)\,dx = -\sin x\,dx \quad \Rightarrow \quad -du = \sin x\,dx \end{aligned}$$

Hence, by the substitution rule,

$$\begin{aligned} \int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx \\ &= -\int \frac{du}{u} \\ &= -\ln|u| + C \\ &= -\ln|\cos x| + C \qquad \text{(substitute back in for } u\text{).} \end{aligned}$$

Since the derivative of this function is exactly the original integrand, we indeed have the correct answer.

Proposition 5.5.8 (Substituion Rule for Definite Integrals)**.** *If $g'(x)$ is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then*

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Proof. Let F be an antiderivative for f. Then $F(g(x))$ is an antiderivative of $f(g(x)) \cdot g'(x)$ by the substitution rule. So, we have

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = F(g(x))\big|_a^b = F(g(b)) - F(g(a))$$

and

$$\int_{g(a)}^{g(b)} f(u)\, du = F(u)\big|_{g(a)}^{g(b)} = F(g(b)) - F(g(a)),$$

whence the definite integrals must be equal. □

Example 5.5.9. Evaluate $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}\, dx$

Let

$$\begin{aligned} u &= g(x) = \frac{1}{x} \\ du &= g'(x)\, dx = -\frac{1}{x^2}\, dx \quad \Rightarrow \quad -du = \frac{1}{x^2}\, dx. \end{aligned}$$

Our new endpoints then become

$$\begin{aligned} u(1) &= g(1) = 1 \\ u(2) &= g(2) = \frac{1}{2}. \end{aligned}$$

Thus, applying the substitution rule, we have

$$\begin{aligned} \int_{x=1}^{x=2} \frac{e^{1/x}}{x^2}\, dx &= \int_{u=1}^{u=1/2} e^u (-du) \\ &= -\int_1^{1/2} e^u\, du \\ &= \int_{1/2}^1 e^u\, du \\ &= e^u\big|_{1/2}^1 \\ &= e^1 - e^{1/2} = e - \sqrt{e}. \end{aligned}$$

Example 5.5.10. Evaluate $\int_0^{1/2} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$

Let

$$\begin{aligned} u &= \arcsin x \\ du &= \frac{1}{\sqrt{1-x^2}}\,dx. \end{aligned}$$

Our new endpoints then become

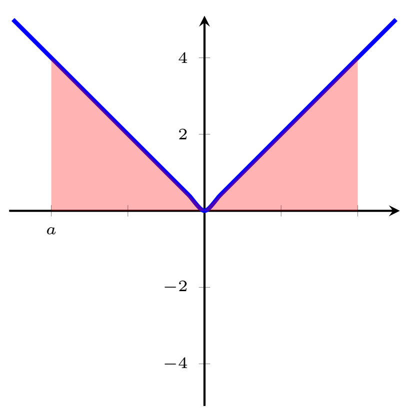
$$\begin{aligned} u(0) \arcsin(0) &= 0 \\ u\left(\frac{1}{2}\right) = \arcsin\left(\frac{1}{2}\right) &= \frac{\pi}{6}. \end{aligned}$$

Thus, applying the substitution rule, we have

$$\begin{aligned} \int_0^{1/2} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx &= \int_0^{\pi/6} u\,du \\ &= \frac{1}{2}u^2\Big|_0^{\pi/6} \\ &= \frac{1}{2}\left(\frac{\pi}{6}\right)^2 - \frac{1}{2}(0)^2 = \frac{\pi^2}{72}. \end{aligned}$$

Recall that a function f is **even** if $f(-x) = f(x)$ **odd** if $f(-x) = -f(x)$, where x is any real number in the domain. The following result tells us about the symmetry of these functions as they relate to definite integrals.

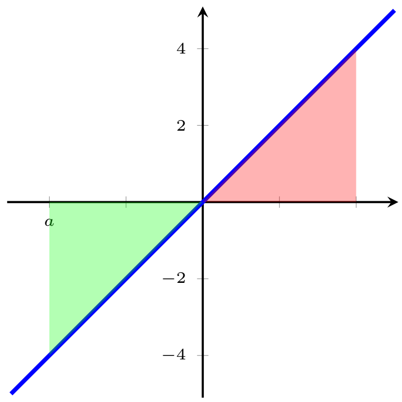
Example 5.5.11. One simple example of an even function is $f(x) = |x|$. Notice that, for some positive real number a, the integral $\int_{-a}^{a} f(x)\,dx$ is represented by the picture below.



Notice that the shaded regions to the left and right of the y-axis are equal, so the area under the curve $y = f(x)$ over the interval $[-a, a]$ is double the area found over the interval $[0, a]$. In other words,

$$\int_{-a}^{a} |x|\,dx = 2\int_0^a |x|\,dx.$$

Example 5.5.12. One simple example of an odd function is $f(x) = x$. Notice that, for some positive real number a, the integral $\int_{-a}^{a} f(x)\,dx$ is represented by the picture below.



Notice that the shaded regions to the left and right of the y-axis are equal, but have opposite sign since area under a curve is "negative". So the area under the curve $y = f(x)$ over the interval $[-a, 0]$ is effectively cancels the area over the interval $[0, a]$. In other words,

$$\int_{-a}^{a} x\,dx = 0.$$

The same sort of symmetry applies in general to even and odd function.

Theorem 5.5.13. *Suppose f is continuous on $[-a, a]$. Then*

1. *If f is even, then* $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$
2. *If f is odd, then* $\int_{-a}^{a} f(x)\,dx = 0.$

Proof. First notice that

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx \tag{5.5.2}$$

For the first integral (with bounds 0 and $-a$), let

$$u = -x,$$
$$du = -dx.$$

Our new bounds are

$$u(0) = 0$$
$$u(-a) = -(-a) = a.$$

Then

$$\begin{aligned}
-\int_0^{-a} f(x)\,dx &= -\int_0^{a} f(-u)(-du) \\
&= \int_0^{a} f(-u)\,du \\
&= \int_0^{a} f(-x)\,dx, && \text{(since } u \text{ was a dummy variable)}
\end{aligned}$$

so from Equation 5.5.2, we can write

$$\int_{-a}^{a} f(x)\,dx = \int_0^{a} f(-x)\,dx + \int_0^{a} f(x)\,dx \tag{5.5.3}$$

1. If f is even, we have $f(-x) = f(x)$, so Equation 5.5.3 yields

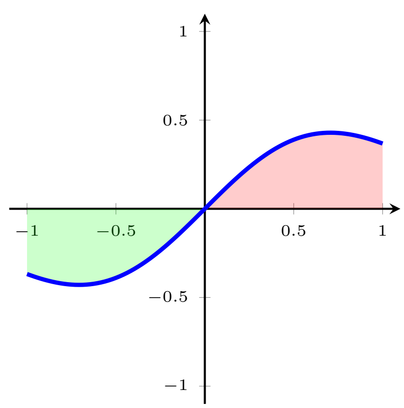
$$\int_{-a}^{a} f(x)\,dx = \int_0^{a} f(-x)\,dx + \int_0^{a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(x)\,dx = 2\int_0^{a} f(x)\,dx.$$

2. If f is odd, we have $f(-x) = -f(x)$, so Equation 5.5.3 yields

$$\int_{-a}^{a} f(x)\,dx = \int_0^{a} f(-x)\,dx + \int_0^{a} f(x)\,dx = -\int_0^{a} f(x)\,dx + \int_0^{a} f(x)\,dx = 0.$$

□

Example 5.5.14. Evaluate the integral $\displaystyle\int_{-1}^{1} xe^{-x^2}\,dx$.



We could preform the substitution, but by first graphing the function with our graphing calculator, we can appeal to the geometry and see that the function is odd. Hence

$$\int_{-1}^{1} xe^{-x^2}\,dx = 0.$$

6 Techniques of Integration

6.1 Integration by Parts

Example 6.1.1 (Warm up)**.** Evaluate $\dfrac{d}{dx}[xe^x]$.

Using the product rule, we have

$$\frac{d}{dx}[xe^x] = e^x + xe^x.$$

Recall that the product rule says

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

In the language of differentials, letting $u = f(x)$ and $v = g(x)$ be differentiable functions, we have that $du = f'(x)\,dx$, $dv = g'(x)\,dx$, and

$$d(uv) = v\,du + u\,dv \quad \Rightarrow \quad u\,dv = d(uv) - v\,du.$$

The following integration technique is then completely analogous to the product rule for derivatives.

Proposition 6.1.2 (Integration by Parts)**.** *Let $u = f(x)$, $v = g(x)$ be differentiable functions. Then*

$$\int u\,dv = uv - \int v\,du. \tag{6.1.1}$$

Remark. Unfortunately, we we typically use u for the substitution method and u, v for Integration by Parts. It should be noted that these u's are wholly unrelated as there is no function substitution happening in Integration by Parts.

The procedure for applying Integration by Parts is as follows:

1. Choose $u = f(x)$, $dv = g'(x)\,dx$ so that $g'(x)\,dx$ is easy to integrate by itself.
2. Find $du = f'(x)$ and $v = \int g'(x)\,dx$.
3. Substitute into Equation 6.1.1 and solve.
4. Apply steps 1-3 again if needed.

Example 6.1.3. Evaluate $\displaystyle\int (e^x + xe^x)\,dx$.

We'll first split this into two separate integrals and solve for the one we already know.

$$\int (e^x + xe^x)\,dx = \int e^x\,dx + \int xe^x\,dx = e^x + \int xe^x\,dx.$$

We'll apply Integration by Parts to solve the rightmost integral. Choose

$$\begin{aligned} u &= x, & du &= dx \\ dv &= e^x\,dx, & v &= e^x. \end{aligned}$$

Substituting into the Integration by Parts formula, we have

$$\int xe^x \, dx = xe^x - \int e^x \, dx = xe^x - e^x + C.$$

Thus the original integral becomes

$$\int (e^x + xe^x) \, dx = e^x + (xe^x - e^x) + C = xe^x + C,$$

which is exactly what we would expect to get from Example 6.1.1.

Example 6.1.4. Evaluate $\int x \cos x \, dx$.

First choose,

$$\begin{aligned} u &= x, & du &= dx, \\ dv &= \cos x \, dx, & v &= \sin x. \end{aligned}$$

Then, substituting into the Integration by Parts formula, we have

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C.$$

What if we had chosen u and v differently in the previous example? Then we would have

$$\begin{aligned} u &= \cos x, & du &= -\sin x \, dx, \\ dv &= x \, dx, & v &= \frac{1}{2}x^2, \end{aligned}$$

and plugging into our Integration by Parts formula gives us

$$\int x \cos x \, dx = \frac{1}{2}x^2 \cos x + \int \frac{1}{2}x^2 \sin x \, dx,$$

and this rightmost integral is even harder to integrate than what we started with.

This suggests to us that, if using integration by parts and one of the functions in the integrand is a polynomial, it might be easiest to choose u to be that polynomial.

Example 6.1.5. Evaluate $\int \arctan x \, dx$.

There aren't a lot of choices for u and dv. Choose

$$u = \arctan x, \qquad du = \frac{1}{1+x^2}\, dx,$$
$$dv = dx, \qquad v = x.$$

Substituting into the Integration by Parts formula, we have

$$\int \arctan x \, dx = x \arctan x - \int \frac{x}{1+x^2}\, dx.$$

How do we evaluate this right-most integral? Use the substitution method. Let

$$w = 1 + x^2$$
$$dw = 2x \, dx \quad \Rightarrow \quad \frac{1}{2}\, dw = x \, dx,$$

So then we have

$$\begin{aligned} \int \arctan x \, dx = x \arctan x - \int \frac{x}{1+x^2}\, dx &= x \arctan x - \frac{1}{2} \int \frac{dw}{w} \\ &= x \arctan x - \frac{1}{2} \ln|w| + C \\ &= x \arctan x - \frac{1}{2} \ln\left(1 + x^2\right) + C. \end{aligned}$$

In some cases, it may be necessary to use integration by parts multiple times.

Example 6.1.6. Evaluate $\int x^2 e^x \, dx$.

Choose

$$u = x^2, \qquad du = 2x \, dx,$$
$$dv = e^x \, dx, \qquad v = e^x.$$

Substituting into the Integration by Parts formula, we have

$$\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx = x^2 e^x - 2 \int x e^x \, dx.$$

We repeat the process for the right-most integral. Choose

$$\tilde{u} = x, \qquad d\tilde{u} = dx,$$
$$d\tilde{v} = e^x \, dx, \qquad \tilde{v} = e^x.$$

Substituting into our Integration by Parts formula, we have

$$\begin{aligned} \int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx &= x^2 e^x - 2 \left(x e^x - \int e^x \, dx \right) \\ &= x^2 e^x - 2\left(x e^x - e^x\right) + C \\ &= x^2 e^x - 2x e^x + 2e^x + C. \end{aligned}$$

When integrating by parts involving a natural logarithm, it's almost always best to let u be the natural logarithm.

Example 6.1.7. Evaluate $\displaystyle\int x^4 \ln x \, dx$.

Since we don't know $\int \ln x \, dx$, choose

$$u = \ln x, \qquad du = \frac{1}{x}\, dx,$$
$$dv = x^4 \, dx, \qquad v = \frac{1}{5}x^5.$$

Substituting into the Integration by Parts formula, we have

$$\begin{aligned} \int x^4 \ln x \, dx &= \frac{1}{5}x^5 \ln x - \int \frac{1}{5}x^4 \, dx \\ &= \frac{1}{5}x^5 \ln x - \frac{1}{25}x^5 + C. \end{aligned}$$

Sometimes you may encounter a situation where you have to repeat the integration by parts and end up with the original integral.

Example 6.1.8. Evaluate $\displaystyle\int e^x \sin x \, dx$.

Choose

$$u = \sin x, \qquad du = \cos x \, dx,$$
$$dv = e^x \, dx, \qquad v = e^x.$$

Substituting into the Integration by Parts formula, we have

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx.$$

Now choose

$$\tilde{u} = \cos x, \qquad d\tilde{u} = -\sin x,$$
$$d\tilde{v} = e^x \, dx, \qquad \tilde{v} = e^x.$$

Again, substituting into the Integration by Parts formula gives us

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx.$$

Putting this all together, we have

$$\begin{aligned} \int e^x \sin x \, dx &= e^x \sin x - \left[e^x \cos x + \int e^x \sin x \, dx \right] \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx \\ \Rightarrow \quad 2\int e^x \sin x \, dx &= e^x \sin x - e^x \cos x + C \\ \Rightarrow \quad \int e^x \sin x \, dx &= \frac{1}{2}\left(e^x \sin x - e^x \cos x\right) + C. \end{aligned}$$

Proposition 6.1.9 (Integration by Parts for Definite Integrals)**.** *Let $u = f(x)$, $v = g(x)$ be differentiable functions and suppose $f'(x)$ and $g'(x)$ are continuous on the interval $[a, b]$. Then*

$$\int_a^b u\,dv = uv\big|_a^b - \int_a^b v\,du. \tag{6.1.2}$$

Example 6.1.10. Evaluate $\displaystyle\int_0^{\pi/2} x \sin x\,dx$.

Choose

$$\begin{aligned} u &= x, & du &= dx,\\ dv &= \sin x\,dx & v &= -\cos x. \end{aligned}$$

Then substituting into Equation 6.1.2, we have

$$\begin{aligned} \int_0^{\pi/2} x \sin x\,dx &= -x\cos x\big|_0^{\pi/2} - \int_0^{\pi/2} -\cos x\,dx\\ &= \int_0^{\pi/2} \cos x\,dx\\ &= \sin x\big|_0^{\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1. \end{aligned}$$

Sometimes we need to use substitution first, and then integration by parts.

Example 6.1.11. Evaluate $\displaystyle\int_0^{\sqrt{\pi}} t^3 \cos\left(t^2\right)\,dt$.

For substitution, choose

$$\begin{aligned} x &= t^2\\ dx &= 2t\,dt \quad \Rightarrow \quad \frac{1}{2}dx = t\,dt. \end{aligned}$$

and our limits become

$$\begin{aligned} x(0) &= 0\\ x(\sqrt{\pi}) &= \pi, \end{aligned}$$

Then

$$\begin{aligned} \int_0^{\sqrt{\pi}} t^3 \cos\left(t^2\right)\,dt &= \int_0^{\sqrt{\pi}} t^2 \cos\left(t^2\right) \cdot t\,dt\\ &= \frac{1}{2}\int_0^{\pi} x\cos x\,dx \end{aligned}$$

and this is the same integral from Example 6.1.4, so we get

$$= \frac{1}{2}\left(x\sin x + \cos x\right)\big|_0^{\pi} = \frac{1}{2}\left(\cos(\pi) - \cos(0)\right) = -1.$$

Example 6.1.12. Evaluate $\displaystyle\int_1^2 \ln(6x+2)\,dx$.

Choose

$$u = \ln(6x+2), \qquad du = \frac{6}{6x+2}\,dx,$$
$$dv = dx, \qquad v = x.$$

Then, substituting into our Integration by Parts formula, we have

$$\begin{aligned}\int_1^2 \ln(6x+2)\,dx &= x\ln(6x+2)\big|_1^2 - \int_1^2 \frac{6x}{6x+2}\,dx\\ &= 2\ln(14) - \ln(8) - \int_1^2 \frac{6x}{6x+2}\,dx.\end{aligned}$$

To solve this remaining integral, we'll need to use substitution, so choose

$$\begin{aligned} w = 6x+2, \quad &\Rightarrow \quad 6x = w-2\\ dw = 6\,dx \quad &\Rightarrow \quad \frac{1}{6}dw = dx,\end{aligned}$$

and it follows that our new limits are

$$\begin{aligned} w(1) &= 6(1)+2 = 8,\\ w(2) &= 6(2)+2 = 14.\end{aligned}$$

So, using substitution,

$$\begin{aligned}\int_1^2 \ln(6x+2)\,dx &= 2\ln(14) - \ln(8) - \int_1^2 \frac{6x}{6x+2}\,dx.\\ &= 2\ln(14) - \ln(8) - \int_8^{14} \frac{w-2}{w}\left(\frac{1}{6}dw\right)\\ &= 2\ln(14) - \ln(8) - \frac{1}{6}\int_8^{14}\left(1-\frac{2}{w}\right)\,dw\\ &= 2\ln(14) - \ln(8) - \frac{1}{6}\left(w - 2\ln w\right)\big|_8^{14}\\ &= 2\ln(14) - \ln(8) - \frac{1}{6}\left(14 - 2\ln(14) - 8 + 2\ln(8)\right)\\ &= \frac{7}{3}\ln(14) - \frac{4}{3}\ln(8) - 1.\end{aligned}$$

6.2 Trigonometric Integrals and Substitutions

6.2.1 Trigonometric Integrals

Before we begin this section, we'll recall the following useful trigonometric identities: The Pythagorean Identities

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1, \\ 1 + \cot^2\theta &= \csc^2\theta, \\ \tan^2\theta + 1 &= \sec^2\theta, \end{aligned}$$

and the Power-Reducing Formulae

$$\begin{aligned} \sin^2\theta &= \frac{1-\cos(2\theta)}{2}, \\ \cos^2\theta &= \frac{1+\cos(2\theta)}{2}. \end{aligned}$$

We will also state the following as fact (the proofs of which are clever and can be found in the book).

$$\int \tan x\,dx = \ln|\sec x| + C, \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C.$$

Example 6.2.1. Evaluate $\int \sin^3 x\,dx$.

We can't apply the substitution rule to this integral as there is no cosine term, and it's not completely obvious how we might integrate this by parts (if it is even possible), so instead we'll use one of the Pythagorean identities above to rewrite the integral.

$$\int \sin^3 x\,dx = \int \sin^2 x \cdot \sin x\,dx = \int (1-\cos^2 x)\sin x\,dx.$$

Now we see we can apply the substitution rule by choosing

$$\begin{aligned} u &= \cos x \\ du &= -\sin x\,dx, \end{aligned}$$

whence

$$\begin{aligned} \int \sin^3 x\,dx &= \int (1-\cos^2 x)\sin x\,dx \\ &= -\int (1-u^2)\,du \\ &= -u + \frac{1}{3}u^3 + C \\ &= -\cos x + \frac{1}{3}\cos^3 x + C \end{aligned}$$

Example 6.2.2. Evaluate $\int \cos^2 x\, dx$.

Using the Pythagorean identity here doesn't simplify anything, so we'll instead use one of the power reducing formulae:

$$\begin{aligned}\int \cos^2 x\, dx &= \int \frac{1+\cos(2x)}{2}\, dx \\ &= \frac{1}{2}\int 1+\cos(2x)\, dx \\ &= \frac{1}{2}\left(x + \frac{1}{2}\sin(2x)\right) + C \qquad \text{(For the } \cos(2x) \text{ part use a substitution } u = 2x\text{)} \\ &= \frac{1}{2}x + \frac{1}{4}\sin(2x) + C\end{aligned}$$

General strategy for handling integrals of the form $\int \sin^m x \cos^n x\, dx$:

(i) If m is odd, save one $\sin x$ factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of $\cos x$. Then use substitution with $u = \cos x$.

(ii) If n is odd, save one $\cos x$ factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of $\sin x$. Then use substitution with $u = \sin x$.

(iii) If m and n are both even, use a power-reducing formula and proceed with either (i) or (ii).

Example 6.2.3. Evaluate $\int \sin^3(5x)\cos^2(5x)\, dx$.

We use the Pythagorean identity $\sin^2(5x) = 1-\cos^2(5x)$, and then apply the substitution $u = \cos(5x)$, $du = -5\sin(5x)\, dx$.

$$\begin{aligned}\int \sin^3(5x)\cos^2(5x)\, dx &= \int \sin(5x)\left(1-\cos^2(5x)\right)\cos^2(5x)\, dx \\ &= -\frac{1}{5}\int (1-u^2)u^2\, du \\ &= -\frac{1}{5}\int u^2 - u^4\, du \\ &= -\frac{1}{15}u^3 + \frac{1}{25}u^5 + C \\ &= -\frac{1}{15}\cos^3(5x) + \frac{1}{25}\cos^5(5x) + C.\end{aligned}$$

Example 6.2.4. Evaluate $\displaystyle\int \sin^4 x\, dx$.

Noting that $\sin^4 x = (\sin^2 x)^2$, we can apply a power-reducing formula.

$$\begin{aligned}
\int \sin^4 x\, dx = \int (\sin^2 x)^2\, dx &= \int \left(\frac{1-\cos(2x)}{2}\right)^2 dx \\
&= \frac{1}{4}\int 1 - 2\cos(2x) + \cos^2(2x)\, dx \\
&= \frac{1}{4}\int 1 - 2\cos(2x) + \left(\frac{1+\cos(4x)}{2}\right) dx \quad \text{(power-reducing formula)} \\
&= \frac{1}{4}\int 1 - 2\cos(2x) + \frac{1}{2} + \frac{1}{2}\cos(4x)\, dx \\
&= \frac{1}{4}\int \frac{3}{2} - 2\cos(2x) + \frac{1}{2}\cos(4x)\, dx \\
&= \frac{1}{4}\left[\frac{3}{2}x - \sin(2x) + \frac{1}{8}\sin(4x)\right] + C \\
&= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C
\end{aligned}$$

General strategy for handling integrals of the form $\displaystyle\int \sec^m x \tan^n x\, dx$:

(i) If m is even, save one $\sec^2 x$ factor and use $\sec^2 x = \tan^2 x + 1$ to express the remaining factors in terms of $\tan x$. Then use substitution with $u = \tan x$.

(ii) If m and n are both odd, save one $\tan x$ factor and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$. Then use substitution with $u = \sec x$.

(iii) If m is odd and n is even, turn and run, or get creative with integration by parts.

Example 6.2.5. Evaluate $\displaystyle\int \sec^4 x \tan^2 x\, dx$.

We use the Pythagorean identity $\sec^2 x = 1 + \tan^2 x$ and then apply the substitution $u = \tan x$, $du = \sec^2 x\, dx$.

$$\begin{aligned}
\int \sec^4 x \tan^2 x\, dx &= \int \sec^2 x (1 + \tan^2 x) \tan^2 x\, dx \\
&= \int (1+u^2)u^2\, du \\
&= \int u^2 + u^4\, du \\
&= \frac{1}{3}u^3 + \frac{1}{5}u^5 + C \\
&= \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C.
\end{aligned}$$

Example 6.2.6. Evaluate $\int \sec^9 x \tan^5 x \, dx$.

We use Pythagorean identity $\tan^2 x = \sec^2 x - 1$ and then apply the substitution $u = \sec x$, $du = \sec x \tan x \, dx$.

$$\begin{aligned}
\int \sec^9 x \tan^5 x \, dx &= \int \sec^9 x \tan x (\tan^4 x) \, dx \\
&= \int \sec^9 x \tan x (\tan^4 x) \, dx \\
&= \int \sec^9 x \tan x (\tan^2 x)^2 \, dx \\
&= \int \sec^9 x \tan x (\sec^2 x - 1)^2 \, dx \\
&= \int u^8 (u^2 - 1)^2 \, du \\
&= \int u^8 (u^4 - 2u^2 + 1) \, du \\
&= \int u^{12} - 2u^{10} + u^8 \, du \\
&= \frac{1}{13} u^{13} - \frac{2}{11} u^{11} + \frac{1}{9} u^9 + C \\
&= \frac{1}{13} \sec^{13} x - \frac{2}{11} \sec^{11} x + \frac{1}{9} \sec^9 x + C.
\end{aligned}$$

Example 6.2.7. Evaluate $\int \sec^3 x \, dx$.

We integrate by parts with

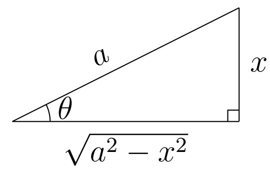
$$\begin{aligned}
u &= \sec x, & du &= \sec x \tan x, \\
dv &= \sec^2 x \, dx, & v &= \tan x.
\end{aligned}$$

So

$$\begin{aligned}
\int \sec^3 x \, dx &= \sec x \tan x - \int \tan^2 x \sec x \, dx \\
&= \sec x \tan x - \int (sec^2 x - 1) \sec x \, dx \\
&= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx \\
&= \sec x \tan x - \ln|\sec x + \tan x| + \int \sec^3 x \, dx \\
2 \int \sec^3 x \, dx &= \sec x \tan x - \ln|\sec x + \tan x| + C \\
\Rightarrow \int \sec^3 x \, dx &= \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln|\sec x + \tan x| + C.
\end{aligned}$$

6.2.2 Trigonometric Substitutions

Suppose we're asked to integrate something of the form $\sqrt{a^2 - x^2}$, where a is some constant real number. None of our techniques so far can be applied directly to this. However, we can think about a, x, and $\sqrt{a^2 - x^2}$ as sitting on a right triangle as follows:



Thinking about it this way, we have that $\sin\theta = \frac{x}{a}$, or rather that $x = a\sin\theta$. Using this substitution, we can rewrite the expression

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2\cos^2\theta} = a\cos\theta \qquad\qquad (\text{since } 0 < \theta < \tfrac{\pi}{2})$$

which is something we *do* know how to integrate. This same procedure gives us a way to handle expressions $\sqrt{x^2 + a^2}$ and $\sqrt{x^2 - a^2}$ as well.

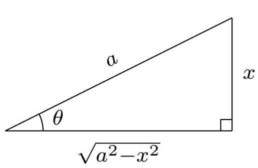
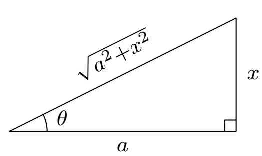
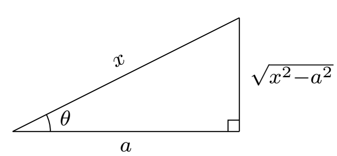
Expression	**Expression**	**Expression**
$\sqrt{a^2 - x^2}$	$\sqrt{a^2 + x^2}$	$\sqrt{x^2 - a^2}$
Substitution	**Substitution**	**Substitution**
$x = a\sin\theta$ $dx = a\cos\theta\, d\theta$	$x = a\tan\theta$ $dx = a\sec^2\theta\, d\theta$	$x = a\sec\theta$ $dx = a\sec\theta\tan\theta\, d\theta$
Trig Identity	**Trig Identity**	**Trig Identity**
$1 - \sin^2\theta = \cos^2\theta$	$\tan^2\theta + 1 = \sec^2\theta$	$\sec^2\theta - 1 = \tan^2\theta$
Reference Triangle	**Reference Triangle**	**Reference Triangle**
		

Table 6.2.1: Trigonometric Substitutions

Example 6.2.8. Perform a trigonometric substitution for the integral $\int \frac{dt}{t^2\sqrt{t^2-16}}$. Evaluate.

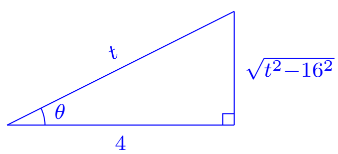
From Table 6.2.1, we use the substitution

$$t = 4\sec\theta$$
$$dt = 4\sec\theta\tan\theta\, d\theta$$

which yields

$$\begin{aligned}\int \frac{dt}{t^2\sqrt{t^2-16}} &= \int \frac{4\sec\theta\tan\theta}{16\sec^2\theta\sqrt{16\sec^2\theta-16}}\, d\theta\\ &= \int \frac{4\sec\theta\tan\theta}{16\sec^2\theta\sqrt{16\tan^2\theta}}\, d\theta\\ &= \frac{1}{16}\int \frac{d\theta}{\sec\theta}\\ &= \frac{1}{16}\int \cos\theta\, d\theta\\ &= \frac{1}{16}\sin\theta + C.\end{aligned}$$

Now, we still need to get our answer back in terms of t. To do this, we fill out the relevant reference triangle



And from this triangle, we see that $\sin\theta = \frac{\sqrt{t^2-16}}{t}$, hence

$$\int \frac{dt}{t^2\sqrt{t^2-16}} = \frac{1}{16}\frac{\sqrt{t^2-16}}{t} + C.$$

Example 6.2.9. Perform a trigonometric substitution for the integral $\int x^3\sqrt{1-x^2}\, dx$. Do not evaluate.

From Table 6.2.1, we use the substitution

$$x = \sin\theta$$
$$dx = \cos\theta\, d\theta$$

which yields

$$\begin{aligned}\int x^3\sqrt{1-x^2}\, dx &= \int \sin^3\theta\sqrt{1-sin^2\theta}\,(\cos\theta)\; d\theta\\ &= \int \sin^3\theta\sqrt{\cos^2\theta}\,(\cos\theta)\; d\theta\\ &= \int \sin^3\theta\cos^2\theta\, d\theta.\end{aligned}$$

Example 6.2.10. Perform a trigonometric substitution for the integral $\int \frac{t^5}{\sqrt{t^2+2}}\, dt$. Do not evaluate.

From Table 6.2.1, we use the substitution

$$\begin{aligned} t &= \sqrt{2}\tan\theta \\ dt &= \sqrt{2}\sec^2\theta\, d\theta \end{aligned}$$

which yields

$$\begin{aligned} \int \frac{t^5}{\sqrt{t^2+2}}\, dt &= \int \frac{(\sqrt{2}\tan\theta)^5}{\sqrt{2\tan^2\theta+2}} \left(\sqrt{2}\sec^2\theta\right)\, d\theta \\ &= \int \frac{8\tan^5\theta\sec^2\theta}{\sqrt{2}\sec\theta}\, d\theta \\ &= 4\sqrt{2}\int \tan^5\theta\sec\theta\, d\theta. \end{aligned}$$

6.3 Partial Fractions

Example 6.3.1 (Warm Up)**.** Rewrite $\dfrac{4}{9(x+4)} + \dfrac{1}{9(2x-1)}$ as a single fraction.

To write as a single fraction, we note that the common denominator will be $9(x+4)(2x-1)$.

$$\begin{aligned}
\frac{4}{9(x+4)} + \frac{1}{9(2x-1)} &= \frac{4(2x-1)}{9(x+4)(2x-1)} + \frac{(x+4)}{9(x+4)(2x-1)} \\
&= \frac{4(2x+1)+(x+4)}{9(x+4)(2x-1)} \\
&= \frac{x}{(x+4)(2x-1)}
\end{aligned}$$

For this integration technique, we'll be going backwards and "uncommonizing" the denominator.

General Strategy for a Partial Fraction Decomposition:

Start with a function $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.

1. If $\deg p(x) \geq \deg q(x)$, we perform polynomial long division to get $f(x) = s(x) + \dfrac{r(x)}{q(x)}$, where $s(x)$ and $r(x)$ are polynomials, and $\deg r(x) < \deg q(x)$.
2. Factor $q(x)$ into a product of linear factors $(ax+b)$ and irreducible quadratic factors (ax^2+bx+c). (You can always do this. Also note that ax^2+bx+c is irreducible if $b^2-4ac<0$.)
3. Rewrite $\dfrac{r(x)}{q(x)}$ as a sum of partial fractions of the form $\dfrac{A}{ax+b}$ or $\dfrac{Ax+B}{ax^2+bx+c}$, where A and B are constants.
4. Solve for the unknown constants in the numerators of your partial fractions.

We'll look at this in cases.

Case I. $q(x)$ factors as a product of distinct linear factors, say

$$q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n).$$

We then have

$$\frac{r(x)}{q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}.$$

Example 6.3.2. Find a partial fraction decomposition of $\dfrac{x}{(x+4)(2x-1)}$.

Since the degree of the numerator is smaller than the degree of the denominator, and the denominator is already factored, we can skip steps 1 and 2 above. Thus we have

$$\frac{x}{(x+4)(2x-1)} = \frac{A}{x+4} + \frac{B}{2x-1}.$$

To solve for A and B, we can clear the denominators by multiplying both sides of the equation by $(x+4)(2x-1)$, leaving us with

$$x = A(2x-1) + B(x+4) = 2Ax - A + Bx + 4B = (2A+B)x + (-A+4B).$$

Two polynomials are equal precisely when their coefficients are equal, so we deduce that

$$\begin{aligned} 2A + B &= 1, \\ -A + 4B &= 0. \end{aligned}$$

And using our favorite method for solving systems of linear equations, we have $A = \frac{4}{9}$ and $B = \frac{1}{9}$. So

$$\frac{x}{(x+4)(2x-1)} = \frac{4}{9(x+4)} + \frac{1}{9(2x-1)},$$

which is exactly what we expected to get given Example 6.3.1.

Example 6.3.3. Use partial fractions to evaluate $\displaystyle\int \frac{5x+1}{(2x+1)(x-1)}\,dx$.

Again we can skip steps 1 and 2 in our partial fraction decomposition, so we get

$$\frac{5x+1}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1},$$

and clearing denominators yields

$$5x + 1 = A(x-1) + B(2x+1) = Ax - A + 2Bx + B = (A+2B)x + (-A+B).$$

so we deduce that

$$\begin{aligned} A + 2B &= 5, \\ -A + B &= 1, \end{aligned}$$

and solving our system, we get that $A = 1$, $B = 2$. So, we can simplify our integral and apply the substitution rule to get

$$\begin{aligned} \int \frac{5x+1}{(2x+1)(x-1)}\,dx &= \int \left(\frac{1}{2x+1} + \frac{2}{x-1}\right)\,dx \\ &= \frac{1}{2}\ln|2x+1| + 2\ln|x-1| + C. \end{aligned}$$

Case II. $q(x)$ factors as a product of repeated linear factors, say

$$q(x) = (a_1x + b_1)(a_2x + b_2)^k.$$

We then have to have an exponent for every power of the repeated factor from 1 to k, so

$$\frac{r(x)}{q(x)} = \frac{A_1}{a_1x + b_1} + \frac{B_1}{(a_2x + b_2)} + \frac{B_2}{(a_2x + b_2)^2} + \cdots + \frac{B_k}{(a_2x + b_2)^k}.$$

Example 6.3.4. Use partial fractions to evaluate $\displaystyle\int \frac{2x+3}{(x+1)^2}\,dx$.

Again we can skip steps 1 and 2 in our partial fraction decomposition, so we get

$$\frac{2x+3}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2},$$

and clearing denominators yields

$$2x + 3 = A(x+1) + B = Ax + (A+B).$$

This gives us the system

$$\begin{aligned} A &= 2, \\ A + B &= 3, \end{aligned}$$

which has solutions $A = 2$, $B = 1$. So we can simplify our integral and apply the substitution rule to get

$$\begin{aligned} \int \frac{2x+3}{(x+1)^2}\,dx &= \int \left(\frac{2}{x+1} + \frac{1}{(x+1)^2}\right) dx \\ &= 2\ln|x+1| - \frac{1}{x+1} + C. \end{aligned}$$

Case III. $q(x)$ factors as a product of distinct irreducible quadratic factors, say

$$q(x) = (a_1x^2 + b_1x + c_1)\cdots(a_nx^2 + b_nx + c_n).$$

We then have to have an exponent for every power of the repeated factor from 1 to k, so

$$\frac{r(x)}{q(x)} = \frac{A_1x + B_1}{a_1x^2 + b_1x + c_1} + \cdots + \frac{A_nx + B_n}{(a_nx^2 + b_nx + c_n)}.$$

Example 6.3.5. Use partial fractions to evaluate $\displaystyle\int \frac{x^3-2x^2+x+1}{x^4+5x^2+4}\,dx$.

Skipping steps 1 and 2, we note that the denominator factors as $(x^2+1)(x^2+1)$, both of which are irreducible quadratics. We then get

$$\frac{x^3-2x^2+x+1}{x^4+5x^2+4} = \frac{x^3-2x^2+x+1}{(x^2+4)(x^2+1)} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{x^2+1},$$

and clearing denominators gives us

$$\begin{aligned} x^3-2x^2+x+1 &= (Ax+B)(x^2+1)+(Cx+D)(x^2+4) \\ &= (A+C)x^3+(B+D)x^2+(A+4C)x+(B+4D), \end{aligned}$$

resulting in the following system of equations

$$\begin{aligned} A+C &= 1, \\ B+D &= -2, \\ A+4C &= 1, \\ B+4D &= 1. \end{aligned}$$

The solutions are $A=1$, $B=-3$, $C=0$, $D=1$. So, by simplifying our integral and applying a substitution rule, we get

$$\begin{aligned} \int \frac{x^3-2x^2+x+1}{x^4+5x^2+4}\,dx &= \int \left(\frac{x-3}{x^2+4}+\frac{1}{x^2+1}\right) dx \\ &= \int \left(\frac{x}{x^2+4}-\frac{3}{x^2+4}+\frac{1}{x^2+1}\right) dx \\ &= \frac{1}{2}\ln|x^2+4| - \frac{3}{2}\arctan\!\left(\frac{x}{2}\right)+\arctan(x)+C. \end{aligned}$$

Case IV. $q(x)$ factors as product of repeated irreducible quadratic factors, say

$$q(x) = (a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)^k.$$

We then have to have an exponent for every power of the repeated factor from 1 to k, so

$$\frac{r(x)}{q(x)} = \frac{A_1x+B_1}{a_1x^2+b_1x+c_1} + \frac{A_2x+B_2}{(a_2x^2+b_2x+c_2)} + \frac{A_3x+B_3}{(a_2x^2+b_2x+c_2)^2} + \cdots + \frac{A_{k+1}x+B_{k+1}}{(a_2x^2+b_2x+c_2)^k}.$$

Example 6.3.6. Use partial fractions to evaluate $\int \frac{x^4+1}{x(x^2+1)^2}\,dx$.

Since the degree of the denominator is larger than that of the denominator, we again get to skip steps 1 and 2. We note also that x^2+1 is an irreducible quadratic. We then get

$$\frac{x^4+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2},$$

and clearing denominators yields

$$\begin{aligned} x^4+1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\ &= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A. \end{aligned}$$

This results in the following system of equations

$$\begin{aligned} A+B &= 1, \\ C &= 0, \\ 2A+B+D &= 0, \\ C+E &= 0, \\ A &= 1, \end{aligned}$$

which has solutions $A=1$, $B=0$, $C=0$, $D=-2$, $E=0$. With this partial fraction decomposition and applying the substitution rule, we get

$$\begin{aligned} \int \frac{x^4+1}{x(x^2+1)^2}\,dx &= \int \left(\frac{1}{x} - \frac{2x}{(x^2+1)^2}\right)dx \\ &= \ln|x| + \frac{1}{x^2+1} + C. \end{aligned}$$

6.3.1 Using a Matrix to Solve Systems of Linear Equations

In the process of performing a partial fraction decomposition, you may end up having to solve for quite a few unknowns, say

$$\begin{aligned} 3x^4+5x^2+17x-9 = (A+2B-3C)x^4+(2A+D)x^3+(B-9C+E)x^2+ \\ +(-A-B-C-D-E)x+(A+C-19E). \end{aligned}$$

By equating coefficients of the polynomial, we get the following system:

$$\begin{array}{cccccccc} A & +2B & -3C & +0D & +0E & = & 3 \\ 2A & +0B & +0C & +D & +0E & = & 0 \\ 0A & +B & -9C & +0D & +E & = & 5 \\ -A & -B & -C & -D & -E & = & 6 \\ A & +0B & +C & +0D & -7E & = & -9 \end{array}$$

This system may be a nightmare to solve, but by dropping the letters and the equals sign, we can rewrite this as an augmented matrix where the left columns correspond to the coefficients of A, B, C, D, E (in order), and the right column is the answer column, like so

$$M = \left(\begin{array}{ccccc|c} 1 & 2 & 3 & 0 & 0 & 3 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -9 & 0 & 1 & 5 \\ -1 & -1 & -1 & -1 & -1 & 6 \\ 1 & 0 & 1 & 0 & -7 & -9 \end{array}\right)$$

By then putting this matrix into *reduced row echelon form* using the $\boxed{\textbf{rref}}$ command, we get

$$\text{rref}(M) = \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & \frac{257}{39} \\ 0 & 1 & 0 & 0 & 0 & \frac{-17}{15} \\ 0 & 0 & 1 & 0 & 0 & \frac{-86}{195} \\ 0 & 0 & 0 & 1 & 0 & \frac{-514}{39} \\ 0 & 0 & 0 & 0 & 1 & \frac{422}{195} \end{array}\right)$$

In this form, we're able to just read off the coefficients in order:

$$A = \tfrac{257}{39}, \quad B = \tfrac{-17}{15}, \quad C = \tfrac{-86}{195}, \quad D = \tfrac{-514}{39}, \quad E = \tfrac{422}{195}.$$

6.4 Integration with Tables and Computer Algebra Systems

Integration can be very hard, but thankfully many integrals, with only a little bit of algebraic manipulation, can be put into a form for which we have known solutions.

Example 6.4.1. Use formula #20 on the table of integrals, $\displaystyle\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$, to evaluate the indefinite integral $\displaystyle\int \frac{\cos x}{\sin^2 x - 9}\,dx$.

We recognize that $u = \sin x$, $du = \cos x\,dx$, and $a = 3$. Then

$$\int \frac{\cos x}{\sin^2 x - 9}\,dx = \frac{1}{6}\ln\left|\frac{\sin x - 3}{\sin x + 3}\right| + C.$$

Of course, half of the challenge of doing this is recognizing the form for the integral to figure out which table formula to use.

Example 6.4.2. Evaluate $\displaystyle\int \frac{\sqrt{2y^2-3}}{y^2}\,dy$ using a table of integrals.

Looking at the table of integrals, we see that #42, $\displaystyle\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u + \sqrt{u^2-a^2}\right| + C$ looks promising. Let $u = \sqrt{2}y$, $du = \sqrt{2}\,dy$, and $a = \sqrt{3}$. Then

$$\begin{aligned}
\int \frac{\sqrt{2y^2-3}}{y^2}\,dy &= \int \frac{2\sqrt{2y^2-3}}{2y^2}\,dy \\
&= \int \frac{2\sqrt{u^2-a^2}}{u^2} \cdot \frac{1}{\sqrt{2}}\,du \\
&= \sqrt{2}\int \frac{\sqrt{u^2-a^2}}{u^2}\,du \\
&= \sqrt{2}\left(-\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u + \sqrt{u^2-a^2}\right|\right) + C \\
&= \sqrt{2}\left(-\frac{\sqrt{2y^2-3}}{\sqrt{2}y} + \ln\left|\sqrt{2}y + \sqrt{2y^2-3}\right|\right) + C \\
&= -\frac{\sqrt{2y^2-3}}{y} + \sqrt{2}\ln\left|\sqrt{2}y + \sqrt{2y^2-3}\right| + C.
\end{aligned}$$

Sometimes, the form may require additional algebraic techniques before applying a substitution.

Example 6.4.3. Evaluate $\displaystyle\int x\sqrt{x^2+2x+5}\,dx$.

Every integral looks like it needs to be of the form $\sqrt{u^2+a^2}$, $\sqrt{u^2-a^2}$, or $\sqrt{a^2-u^2}$. So, we need to complete the square.

$$x^2+2x+5 = x^2+2x+1+5-1 = (x+1)^2+4$$

So, with $u = x+1$, $du = dx$, and $a = 2$, we have

$$\begin{aligned}\int x\sqrt{x^2+2x+5}\,dx &= \int (u-1)\sqrt{u^2+a^2}\,du \\ &= \int u\sqrt{u^2+a^2}\,du - \int \sqrt{u^2+a^2}\,du\end{aligned}$$

The left-hand integral can be approached with the substitution $t = u^2+a^2$ and $dt = 2u\,du$,

$$\int u\sqrt{u^2+a^2}\,du = \int \frac{1}{2}\sqrt{t}\,dt = \frac{1}{3}t^{3/2}+C = \frac{1}{2}(u^2+a^2)^{3/2}+C,$$

and the right-hand integral can be approached with equation #21,

$$\int \sqrt{u^2+a^2}\,du = \frac{u}{2}\sqrt{u^2+a^2}+\frac{a^2}{2}\ln\left(u+\sqrt{u^2+a^2}\right)+C.$$

Putting it all together, we get

$$\begin{aligned}&\int u\sqrt{u^2+a^2}\,du - \int \sqrt{u^2+a^2}\,du \\ &= \frac{1}{2}(u^2+a^2)^{3/2} - \frac{u}{2}\sqrt{u^2+a^2} - \frac{a^2}{2}\ln\left(u+\sqrt{u^2+a^2}\right)+C \\ &= \frac{1}{2}(x^2+2x+5)^{3/2} - \frac{x+1}{2}\sqrt{x^2+2x+5} - 2\ln\left(x+1+\sqrt{x^2+2x+5}\right)+C.\end{aligned}$$

There are several CAS (computer algebraic systems) that can perform algebraic integration and you, as a student, likely have free acces to at least one of them: Mathematica, Maple, Sage, etc. These must be approached with caution, for there are some idiosyncracies of which you should be aware. For example, asking Mathematica (or the free online WolframAlpha) to find $\int x\sqrt{x^2+2x+5}\,dx$, we get

$$\int x\sqrt{x^2+2x+5}\,dx = \frac{1}{6}\sqrt{x^2+2x+5}\left(2x^2+x+7\right) - 2\sinh^{-1}\left(\frac{x+1}{2}\right)+C.$$

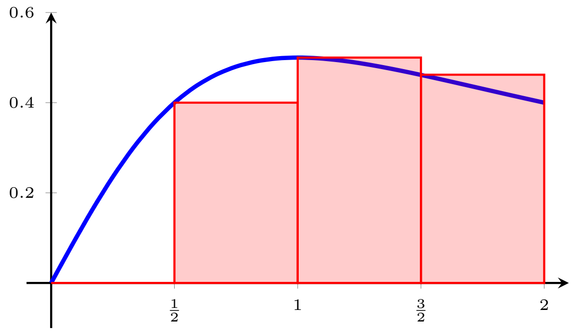
While correct, this isn't a particularly useful form, because hyperbolic trigonometric functions (especially their inverses like $\sinh^{-1}$) are quite uncommon in practice. This solution ends up being equivalent to ours via the identity $\sinh^{-1}x = \ln(x+\sqrt{x^2+1})$, which is arguably in a much nicer form.

As well, it's not uncommon for CAS to return integrals without the $+C$ (they return just a particular antiderivative), or to forget the absolute value signs in the logarithms.

6.5 Approximate Integration

Back in Chapter 5, we saw ways to approximate indefinite integrals using left, right, and middle Riemann sums. In this section, we'll see two new ways to approximate the area under a curve in ways that are equally about as easy to implement.

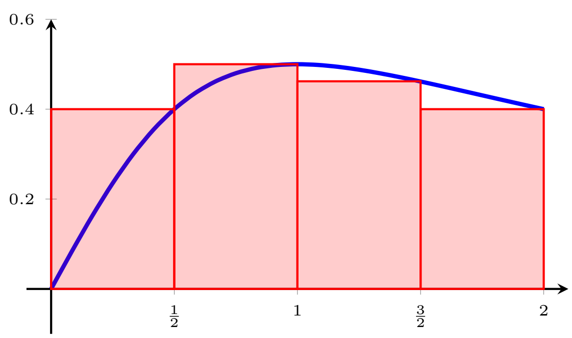
Example 6.5.1. Using left Riemann sums, approximate $\displaystyle\int_0^2 \frac{x}{1+x^2}\,dx$ with four rectangles.



Recall that the left Riemann sum L_4 is given by the formula

$$\begin{aligned} L_4 = \sum_{i=1}^{4} f(x_{i-1})\Delta x &= \sum_{i=1}^{4} f(x_{i-1}) \left(\frac{2-0}{4}\right) \\ &= f(0)(0.5) + f(0.5)(0.5) + f(1)(0.5) + f(1.5)(0.5) \\ &= 0(0.5) + (0.4)(0.5) + (0.5)(0.5) + (0.461538)(0.5) \\ &= 0.680769. \end{aligned}$$

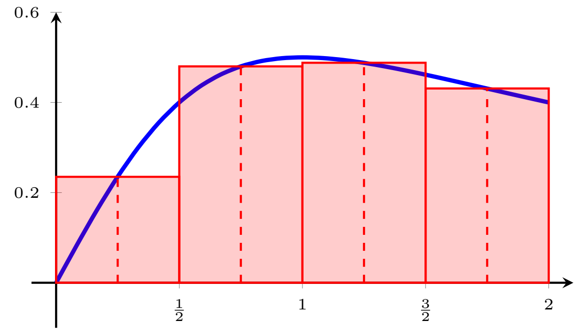
Example 6.5.2. Using right Riemann sums, approximate $\displaystyle\int_0^2 \frac{x}{1+x^2}\,dx$ with four rectangles.



Recall that the right Riemann sum R_4 is given by the formula

$$\begin{aligned} L_4 = \sum_{i=1}^{4} f(x_i)\Delta x &= \sum_{i=1}^{4} f(x_i) \left(\frac{2-0}{4}\right) \\ &= f(0.5)(0.5) + f(1)(0.5) + f(0.5)(0.5) + f(2)(0.5) \\ &= (0.4)(0.5) + (0.5)(0.5) + (0.461538)(0.5) + (0.4)(0.5) \\ &= 0.880769. \end{aligned}$$

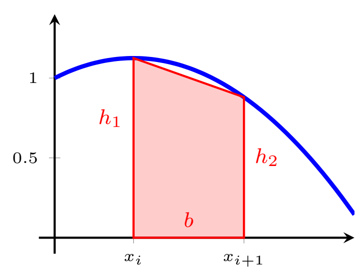
Example 6.5.3. Using middle Riemann sums (AKA, the midpoint rule), approximate $\int_0^2 \frac{x}{1+x^2}\,dx$ with four rectangles.



Recall that the middle Riemann sum M_4 is given by the formula

$$\begin{aligned} M_4 = \sum_{i=1}^{4} f\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x = \sum_{i=1}^{4} f\left(\frac{x_i + x_{i-1}}{2}\right)\left(\frac{2-0}{4}\right) &= f(0.25)(0.5) + f(0.75)(0.5) + f(1.25)(0.5) + f(1.75)(0.5) \\ &= (0.235294)(0.5) + (0.48)(0.5) + (0.487805)(0.5) + (0.430769)(0.5) \\ &= 0.816934. \end{aligned}$$

This whole time we've been approximating with rectangles, but that's only because it's a convenient shape for which we already know the area. In a similar vain, we can approximate with trapezoids as well:

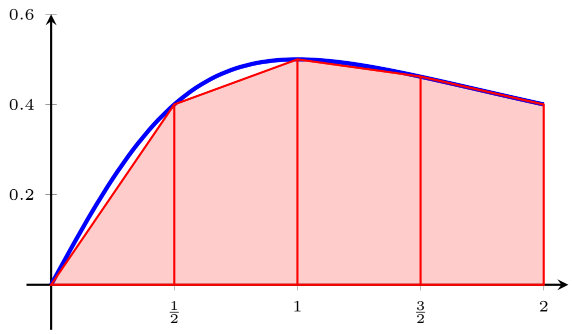


The area of a trapezoid with base b and heights h_1, h_2 is given by $A = \frac{1}{2}(h_1 + h_2)b$. In terms of our function, the area is given by $\frac{1}{2}(f(x_i) + f(x_{i+1}))\Delta x$. If we sum over a n trapezoidal areas of this form, we get an approximation for the area under the curve on the interval $[a, b]$. This gives us

Proposition 6.5.4 (Trapezoid Rule)**.**

$$\begin{aligned} \int_a^b f(x) \approx T_n &= \frac{1}{2}\left(f(x_0) + f(x_1)\right)\Delta x + \frac{1}{2}\left(f(x_1) + f(x_2)\right)\Delta x + \cdots + \frac{1}{2}\left(f(x_{n-1}) + f(x_n)\right)\Delta x \\ &= \frac{1}{2}\Delta x\Big[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\Big]. \end{aligned}$$

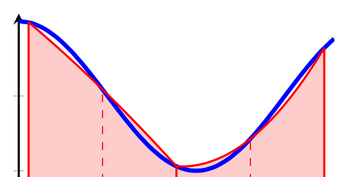
Example 6.5.5. Using the Trapezoid Rule, approximate $\int_0^2 \frac{x}{1+x^2}\,dx$ with four trapezoids.



By the Trapezoid Rule gives the approximation

$$\begin{aligned} T_4 &= \frac{1}{2}\left(\frac{2-0}{4}\right)\Big[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)\Big] \\ &= \frac{1}{2}(0.5)\Big[f(0)+2f(0.5)+2f(1)+2f(1.5)+f(2)\Big] \\ &= \frac{1}{2}(0.5)\Big[0+2(0.4)+2(0.5)+2(0.461538)+(0.4)\Big] \\ &= 0.780769. \end{aligned}$$

It is a general fact that for any three distinct, non-collinear points, we can find a unique parabola that passes through all three. This means that, if we break our interval up into an *even number* of smaller intervals $[x_i, x_{i+1}]$, we can find a parabola passing through the points $f(x_{i-1})$, $f(x_i)$, and $f(x_{i+1})$ whose area over the interval $[x_{i-1}, x_{i+1}]$ approximates the area of the curve over this same interval. Visually,

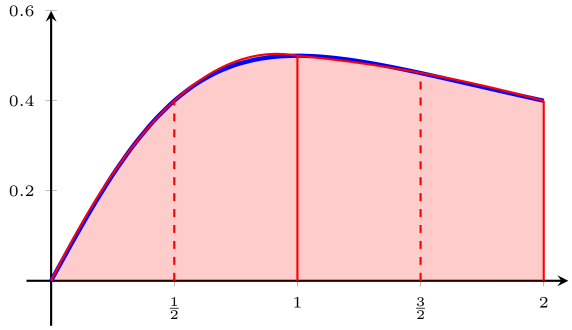


This fact leads us to another approximation method that is easy to implement (but whose derivation is less straightforward – see your book for details). Note that the coefficient pattern is $1, 4, 2, 4, 2, \ldots, 2, 4, 1$.

Proposition 6.5.6 (Simpson's Rule)**.** *For even n,*

$$\int_a^b f(x)\,dx \approx S_n = \frac{1}{3}\Delta x\Big[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+4f(x_5)+\cdots+4f(x_{n-1})+f(x_n)\Big].$$

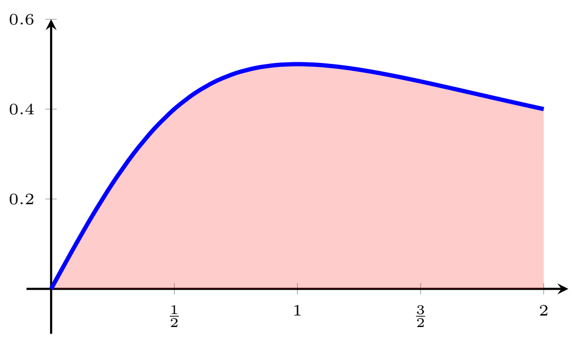
Example 6.5.7. Using Simpson's rule, approximate $\int_0^2 \frac{x}{1+x^2}\,dx$ with four intervals.



Applying Simpson's Rule, we have

$$\begin{aligned}
S_4 &= \frac{1}{3}\Delta x\Big[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+f(x_4)\Big] \\
&= \frac{1}{3}\left(\frac{2-0}{4}\right)\Big[f(0)+4f(0.5)+2f(1)+4f(1.5)+f(2)\Big] \\
&= \frac{1}{3}(0.5)\left[0+4(0.4)+2(0.5)+4(0.461538)+(0.4)\right] \\
&= 0.807692.
\end{aligned}$$

Example 6.5.8. Using the Fundamental Theorem of Calculus, compute $\int_0^2 \frac{x}{1+x^2}\,dx$ exactly. Compare this answer with the previous approximations. Which is most accurate?



Using the substitution

$$\begin{aligned}
u &= 1+x^2 \\
du &= 2x\,dx \\
u(0) &= 1, \\
u(2) &= 5,
\end{aligned}$$

we get

$$\int_0^2 \frac{x}{1+x^2}\,dx = \int 1^5 \frac{1}{2}\cdot\frac{du}{u} = \frac{1}{2}\ln(5) - \frac{1}{2}\ln(1) = \frac{1}{2}\ln(5) \approx 0.804719$$

Method	**Approximation**	**Error**
Left Riemann Sum	0.680769	0.123950
Right Riemann Sum	0.880769	0.076050
Middle Riemann Sum	0.816934	0.012215
Trapezoid Rule	0.780769	0.023950
Simpson's Rule	0.807692	0.002973
Exact Value	0.804719	0.000000

Simpson's rule is by far the most accurate approximation.

6.6 Improper Integrals

Definition. The integral $\int_a^b f(x)\,dx$ is **improper** if either the integrand is infinite on the interval $[a,b]$ or if the interval is infinite.

As with just every other time we've encountered the infinite in this class, we'll use limits to handle it.

6.6.1 Type I: Infinite Interval

Definition (Improper Integrals of Type I)**.**

(a) If $\int_a^t f(x)\,dx$ exists for every $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx,$$

provided this limit exists.

(b) If $\int_t^b f(x)\,dx$ exists for every $t \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx,$$

provided this limit exists.

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists, and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent for any real number a, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx.$$

Example 6.6.1. Evaluate $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$.

We cannot actually evaluate this integral at infinity. However, we can evaluate it on the interval $[1,t]$ and let $t\to\infty$. So

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}\,dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx\\ &= \lim_{t\to\infty}\left(-\frac{1}{x}\right)\Big|_1^t\\ &= \lim_{t\to\infty}\left(-\frac{1}{t}+1\right) = 1.\end{aligned}$$

Example 6.6.2. Determine whether the integral converges or diverges: $\displaystyle\int_3^\infty \frac{1}{(x-2)^{3/2}}\,dx$.

We'll need the substitution $u = x-2$ and $du = dx$. So, then $u(3) = 1$ and $u(b) \to \infty$ as $b \to \infty$.

$$\begin{aligned}
\int_3^\infty \frac{1}{(x-2)^{3/2}}\,dx = \lim_{b\to\infty}\int_3^b \frac{1}{(x-2)^{3/2}}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{u^{3/2}}\,du \\
&= \lim_{b\to\infty}\int_1^b u^{-3/2}\,du \\
&= \lim_{b\to\infty} -2u^{-1/2}\Big|_1^b \\
&= \lim_{b\to\infty} -2b^{-1/2}+2 \\
&= 0+2 = 2.
\end{aligned}$$

The integral converges.

Example 6.6.3. Evaluate $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ if it converges. Otherwise state that it does not converge.

$$\begin{aligned}
\int_1^\infty \frac{1}{x}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx \\
&= \lim_{b\to\infty} \ln x\Big|_1^b \\
&= \lim_{b\to\infty} \ln b - \ln 1 \\
&= \infty + 0 = \infty.
\end{aligned}$$

The integral diverges.

Example 6.6.4. For which values of p does the integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converge?

As we saw in Example 6.6.3, when $p = 1$, the integral does not converge. So let's see about the case where $p \neq 1$.

$$\begin{aligned}
\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx \\
&= \lim_{b\to\infty}\int_1^b x^{-p}\,dx \\
&= \lim_{b\to\infty} \frac{1}{-p+1}x^{(-p+1)}\Big|_1^b \\
&= \lim_{b\to\infty} \frac{1}{-p+1}\left(b^{(-p+1)}-1\right).
\end{aligned}$$

We see that $\lim\limits_{b\to\infty} b^{(-p+1)}$ converges precisely when $-p+1 < 0$, which is precisely when $p > 1$. Thus the integral converges for all $p > 1$.

Example 6.6.5. Evaluate $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,dx$.

We'll first approach the indefinite integral with the substitution $u = -x^2$, $du = -2x\,dx$:

$$\int xe^{-x^2}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C.$$

Thus we can break up the integral at some arbitrary point in the interval $(-\infty, \infty)$, say at $x = 0$.

$$\begin{aligned}
\int_{-\infty}^{\infty} xe^{-x^2}\,dx &= \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_{0}^{\infty} xe^{-x^2}\,dx \\
&= \lim_{a\to-\infty}\int_{a}^{0} xe^{-x^2}\,dx + \lim_{b\to\infty}\int_{0}^{b} xe^{-x^2}\,dx \\
&= \lim_{a\to-\infty}\left(-\frac{1}{2} + \frac{1}{2}e^{-a^2}\right) + \lim_{b\to\infty}\left(-\frac{1}{2}e^{-b^2} + \frac{1}{2}\right) \\
&= \left(-\frac{1}{2} + 0\right) + \left(0 + \frac{1}{2}\right) = 0.
\end{aligned}$$

6.6.2 Type II: Discontinuous Integrand

Definition (Improper Integrals of Type II)**.**

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_{a}^{b} f(x)\,dx = \lim_{t\to b^-}\int_{a}^{t} f(x)\,dx$$

provided this limit exists.

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_{a}^{b} f(x)\,dx = \lim_{t\to a^+}\int_{t}^{b} f(x)\,dx$$

provided this limit exists.

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, a real number in (a, b), and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_{a}^{b} f(x)\,dx = \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx.$$

Example 6.6.6. Evaluate $\displaystyle\int_0^3 \frac{1}{(x-2)^2}\,dx$.

Notice that the integrand is undefined at $x = 2$, so we'll need to split it up here.

$$\begin{aligned}
\int_0^3 \frac{1}{(x-2)^2}\,dx &= \int_0^2 \frac{1}{(x-2)^2}\,dx + \int_2^3 \frac{1}{(x-2)^2}\,dx \\
&= \lim_{b\to 2^-} \int_0^b \frac{1}{(x-2)^2}\,dx + \lim_{a\to 2^+} \int_a^3 \frac{1}{(x-2)^2}\,dx.
\end{aligned}$$

Now, with the substitution $u = x-2$ and $du = dx$, we have that $u(0) = -2$, $u(2) = 0$ and $u(3) = 1$, so

$$\begin{aligned}
&= \lim_{b\to 0^-} \int_{-2}^b \frac{1}{u^2}\,du + \lim_{a\to 0^+} \int_a^1 \frac{1}{u^2}\,du \\
&= \lim_{b\to 0^-} \left(-b^{-1} + (-2)^2\right) + \lim_{a\to 0^+} \left(-1 + a^1\right).
\end{aligned}$$

Since neither of these limits converge, the integral is divergent.

Example 6.6.7. Evaluate $\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$.

The integrand is undefined at $x = 1$, so we'll need to split it up here.

$$\begin{aligned}
\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx &= \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx \\
&= \lim_{b\to 1^-} \int_0^b \frac{1}{\sqrt[3]{x-1}}\,dx + \lim_{a\to 1^+} \int_a^9 \frac{1}{\sqrt[3]{x-1}}\,dx.
\end{aligned}$$

Now, with the substitution $u = x-1$ and $du = dx$, we have that $u(0) = -1$, $u(1) = 0$ and $u(9) = 8$, so

$$\begin{aligned}
&= \lim_{b\to 0^-} \int_{-1}^b u^{-1/3}\,dx + \lim_{a\to 0^+} \int_a^8 u^{-1/3}\,dx \\
&= \lim_{b\to 0^-} \left[\frac{3}{2}u^{2/3}\right]_{-1}^b + \lim_{a\to 0^+} \left[\frac{3}{2}u^{2/3}\right]_a^8 \\
&= \lim_{b\to 0^-} \left(\frac{3}{2}b^{2/3} - \frac{3}{2}\right) + \lim_{a\to 0^+} \left(6 - \frac{3}{2}a^{2/3}\right) \\
&= -\frac{3}{2} + 6 = \frac{9}{2}.
\end{aligned}$$

6.6.3 A Comparison Test for Improper Integrals

Before trying to evaluate or approximate an improper integral, it's important to know whether or not the integral even converges.

Theorem 6.6.8 (Comparison Tests)**.** *Suppose f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.*

1. *If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.*
2. *If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.*

Example 6.6.9. Does $\displaystyle\int_3^\infty \frac{1}{x^2 \ln x}\,dx$ converge or diverge?

Notice that for $x \geq 3$, we have

$$\begin{aligned} 1 &\leq \ln x \\ x^2 &\leq x^2 \ln x \\ \frac{1}{x^2} &\geq \frac{1}{x^2 \ln x}. \end{aligned}$$

So, we'll apply our comparison test with $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x^2 \ln x}$.

We know from Example 6.6.4 that $\int_3^\infty f(x)\,dx$ converges, so then it must be that $\int_3^\infty g(x)\,dx$ converges as well.

Example 6.6.10. Does $\displaystyle\int_1^\infty \frac{2 + e^{-x}}{x}\,dx$ converge or diverge?

Since $e^{-x} \geq 0$ for all x, we have

$$\frac{2 + e^{-x}}{x} \geq \frac{2}{x}.$$

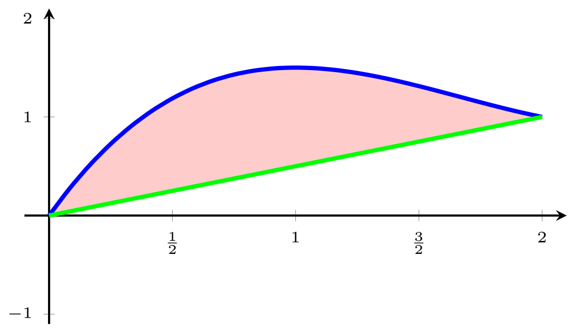
So, we'll apply our comparison test with $f(x) = \frac{2+e^{-x}}{x}$ and $g(x) = \frac{2}{x}$.

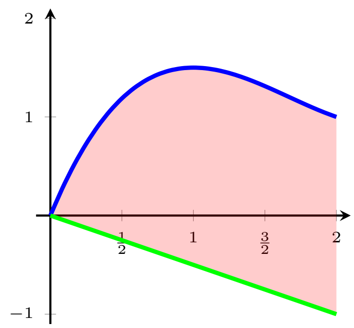
We know from Example 6.6.3 that $\int_1^\infty g(x)\,dx$ diverges, so then it must be that $\int_1^\infty f(x)\,dx$ diverges also.

7 Applications of Integration

7.1 Area Between Curves

Recall that the area under the curve $y = f(x)$ on the interval $[a, b]$ is given by the definite integral $\int_a^b f(x)\,dx$. What if we wanted to find the area between two curves? Consider both of the following figures below.



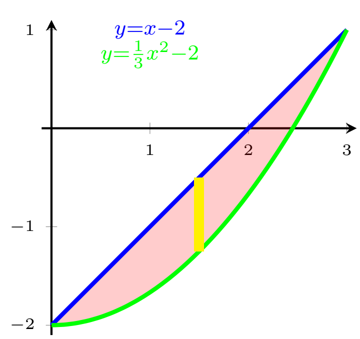


We suspect that the area between the curves is just the difference of the integrals. Indeed, this is the case. If we consider formally the Riemann sums, we have that at each marked point x^* each rectangle has height $f(x^*) - g(x^*)$ (when $f(x) \geq g(x)$), and the sum of all these gives us exactly the following

The **area between the curves** $y = f(x)$ **and** $y = g(x)$ on the interval $[a, b]$ is

$$A = \int_a^b |f(x) - g(x)|\,dx.$$

Example 7.1.1. Find the area between the curves $y = x - 2$ and $y = \frac{x^2}{3} - 2$



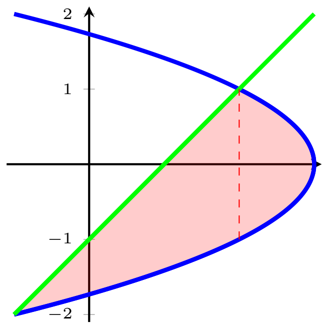
First we find our limits of integration by setting the two functions equal to one another and solving for x:

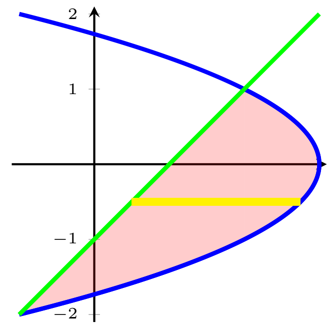
$$\begin{aligned} x - 2 &= \frac{x^2}{3} - 2 \\ \frac{x^2}{3} - x &= 0 \\ \frac{1}{3}x(x-3) &= 0 \\ \Rightarrow \quad x &= 0, 3 \end{aligned}$$

Since $y = x - 2$ is the "top" function, we have

$$\begin{aligned} \int_0^3 \left((x-2) - \left(\frac{x^2}{3} - 2\right)\right) dx &= \int_0^3 \left(x - \frac{x^2}{3}\right) dx \\ &= \left[\frac{x^2}{2} - \frac{x^3}{9}\right]_0^3 = \frac{3}{2}. \end{aligned}$$

Example 7.1.2. Find the area between the curves $y^2 = 3 - x$ and $y = x - 1$





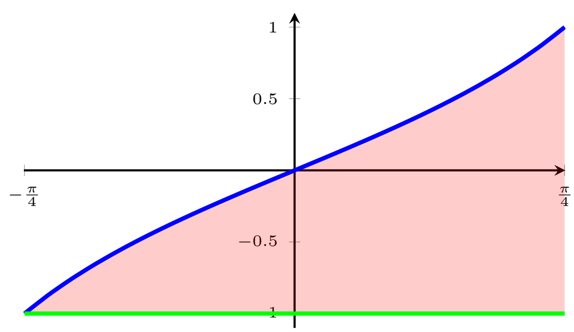
Notice in the image on the left that we would have to have two separate integrals, because the "top" function would change at $x = \frac{1}{2}$. Instead, we'll draw horizontal rectangles and just integrate with respect to y, since the "top" and "bottom" functions do not change in this direction. This also saves us from the hassle of integrating square roots of functions. We proceed by solving each equation for x (in terms of y) and setting them equal to find the points of intersection:

$$\begin{aligned} 3 - y^2 &= y + 1 \\ y^2 + y - 2 &= 0 \\ (y+2)(y-1) &= 0 \\ \Rightarrow \quad y &= -2, 1. \end{aligned}$$

So, the area between these two curves is

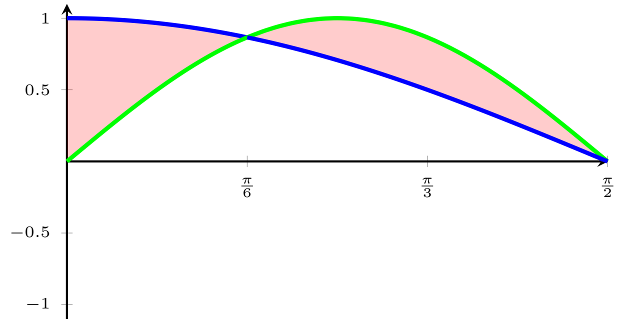
$$\begin{aligned} \int_{-2}^{1} \left((y+1) - \left(3 - y^2\right)\right)\, dx &= \int_{-2}^{1} \left(-y^2 - y + 2\right)\, dx \\ &= \left[-\frac{1}{3}y^3 - \frac{1}{2}y^2 + 2y\right]_{-2}^{1} = \frac{9}{2}. \end{aligned}$$

Example 7.1.3. Find the area of the region bounded by $y = \tan x$, $y = -1$, and $x = \frac{\pi}{4}$.



$$\begin{aligned} \int_{-\pi/4}^{\pi/4} (\tan x - (-1))\, dx &= \int_{-\pi/4}^{\pi/4} (\tan x + 1)\, dx \\ &= \left[\ln|\sec x| + x\right]_{-\pi/4}^{\pi/4} \\ &= \frac{\pi}{2}. \end{aligned}$$

Example 7.1.4. Find the area between the two curves $y = \cos x$ and $y = \sin 2x$ on the interval $[0, \frac{\pi}{2}]$.



We have to consider two separate areas. Note that the curves intersect at $x = \frac{\pi}{6}$.

$$\begin{aligned} A_1 &= \int_0^{\pi/6} (\cos x - \sin(2x))\ dx \\ &= \left[\sin x + \frac{1}{2}\cos(2x)\right]_0^{\pi/6} \\ &= \frac{1}{4}. \end{aligned}$$

and

$$\begin{aligned} A_2 &= \int_{\pi/6}^{\pi/2} (\sin(2x) - \cos x)\ dx \\ &= \left[-\frac{1}{2}\cos(2x) - \sin x\right]_{\pi/6}^{\pi/2} \\ &= \frac{1}{4}. \end{aligned}$$

And so the entire area is $A = A_1 + A_2 = \frac{1}{2}$.

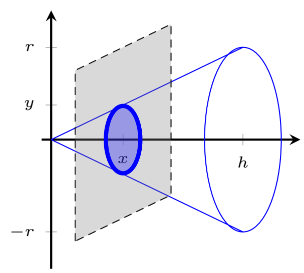
7.2 Volumes

Just as we defined area as summing up a bunch of infinitestimal lengths, we can do the same with volumes by summing up a bunch of infinitesimal cross-sectional areas. Formally, we have the following definition.

Definition. Let S be a solid lying between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x (through x and perpendicular to the x-axis), is $A(x)$, where A is an integrable function, then the **volume** of S is

$$V = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} A(x_i^*)\Delta x_i = \int_a^b A(x)\, dx$$

Example 7.2.1. What is the volume of a right circular cone of height h and radius r?



Notice that the cross-sectional area through x is a circle of radius y, so the area is $A(x) = \pi y^2$. In order to integrate A with respect to x, we need to write y as a function of x. Notice that by similar triangles,

$$\frac{r}{h} = \frac{y}{x} \quad \Rightarrow y = \frac{rx}{h}.$$

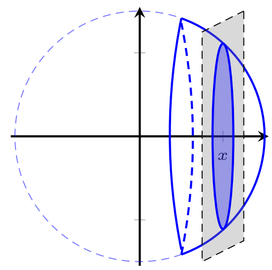
Thus the volume is given by

$$\begin{aligned} V = \int_0^h \pi y^2\, dx &= \int_0^h \pi \left(\frac{rx}{h}\right)^2 dx \\ &= \frac{\pi r^2}{h^2} \int_0^h x^2\, dx \\ &= \frac{\pi r^2}{h^2} \left[\frac{1}{3}h^3 - 0\right] \\ &= \frac{1}{3}\pi r^2 h. \end{aligned}$$

This is exactly the volume of a cone that we all remember from the countless conical tank related rates problems.

The previous example affirms that our strategy for finding the volume is indeed correct. So now we can use it to find the volume of objects whose volumes don't know *a priori*.

Example 7.2.2. Find the volume of a spherical cap with radius r and height h.



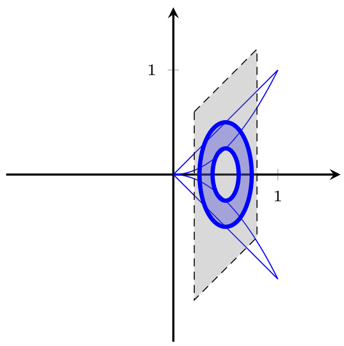
At each point x, the corresponding cross-section has radius $y = \sqrt{r^2 - x^2}$. This means that each cross-sectional area is

$$A(x) = \pi y^2 = \pi(r^2 - x^2).$$

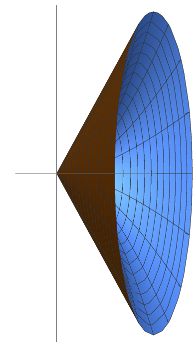
Since our spherical cap has height h, we integrate from $x = r - h$ to r, which gives us

$$\begin{aligned}
V &= \int_{r-h}^{r} A(x)\,dx \\
&= \int_{r-h}^{r} \pi(r^2 - x^2)\,dx \\
&= \left[\pi r^2 x - \frac{\pi}{3}x^3\right]_{r-h}^{r} \\
&= \pi r^3 - \frac{\pi}{3}r^3 - \pi r^2(r-h) + \frac{\pi}{3}(r-h)^3 \\
&= \pi r^3 - \frac{\pi}{3}r^3 - \pi r^3 + \pi r^2 h + \frac{\pi}{3}r^3 - \pi r^2 h + \pi r h^2 - \frac{\pi}{3}h^3 \\
&= \frac{\pi}{3}h^2(3r - h).
\end{aligned}$$

Example 7.2.3. Find the volume of the solid obtained by rotating the region bounded by the curves $y = x$ and $y = x^2$ about the x-axis.



(a) Cross-sectional Area



(b) Revolved Solid

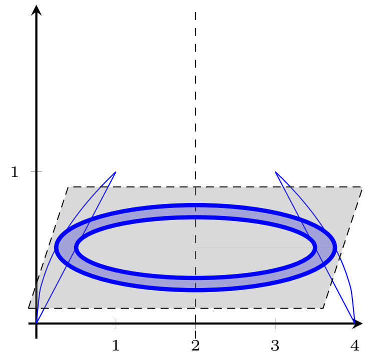
At each point x, the corresponding cross-section is a washer with inner radius x^2 and outer radius x. This means that each cross-sectional area is

$$A(x) = \pi x^2 - \pi(x^2)^2.$$

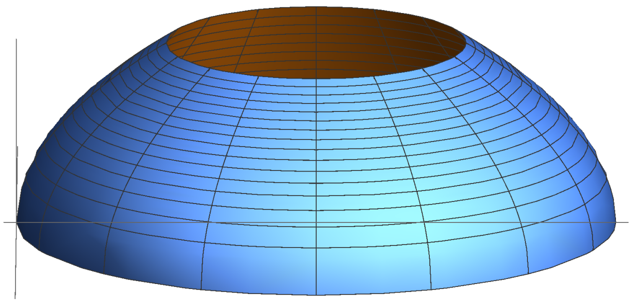
Since $y = x$ and $y = x^2$ intersect at both $x = 0$ and $x = 1$, we integrate from $x = 0$ to 1, which gives us

$$\begin{aligned} V &= \int_0^1 A(x)\,dx \\ &= \int_0^1 \pi x^2 - \pi x^4\,dx \\ &= \left[\frac{\pi}{3}x^3 - \frac{\pi}{5}x^5\right]_0^1 \\ &= \frac{\pi}{3} - \frac{\pi}{5} \\ &= \frac{2\pi}{15}. \end{aligned}$$

Example 7.2.4. Find the volume of a solid obtained by rotating the region bounded by the curves $y = x$ and $y = \sqrt{x}$ about the line $x = 2$.



(a) Cross-sectional Area



(b) Revolved Solid

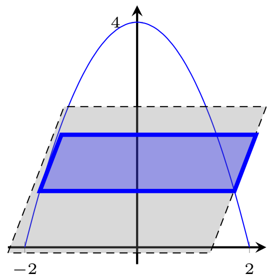
Notice that each horizontal cross section (now we're looking to integrate with respect to y) is a washer with inner radius $2 - y$ and outer radius $2 - y^2$, and thus the area is given by

$$A(y) = \pi(2 - y^2)^2 - \pi(2 - y)^2 = 4\pi - 4\pi y^2 + \pi y^4 - 4\pi + 4\pi y - \pi y^2 = \pi y^4 - 5\pi y^2 + 4\pi y.$$

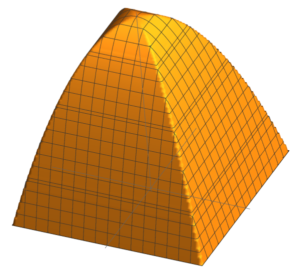
Since $y = x$ and $y = \sqrt{x}$ intersect at both $y = 0$ and $y = 1$, we integrate from $y = 0$ to 1, which gives us

$$\begin{aligned} V &= \int_0^1 A(y)\, dy \\ &= \int_0^1 \pi y^4 - 5\pi y^2 + 4\pi y\, dy \\ &= \left[\frac{\pi}{5} y^5 - \frac{5\pi}{3} y^3 + 2\pi y^2\right]_0^1 \\ &= \frac{\pi}{5} - \frac{5\pi}{3} + 2\pi = \frac{8\pi}{15}. \end{aligned}$$

Example 7.2.5. Find the volume of the solid bounded by $y = 4 - x^2$ and the x-axis whose horizontal cross-sections are all squares.



(a) Cross-sectional Area



(b) Solid Region

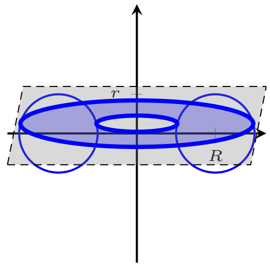
For each y, cross-sectional area here is a square with side length $2x = 2\sqrt{4-y}$. The area function is thus

$$A(y) = (2x)^2 = 4(4-y) = 16 - 4y.$$

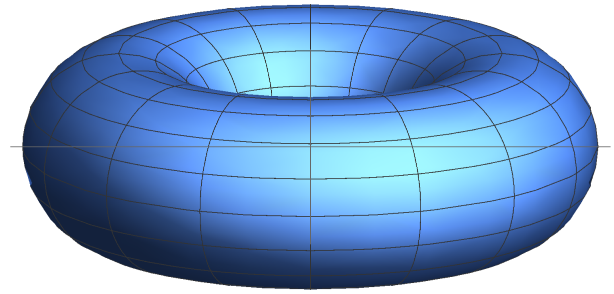
Notice that we'll integrate from $y = 0$ to 4. Thus the volume of this solid is given by

$$\begin{aligned} V &= \int_0^4 A(y)\,dy \\ &= \int_0^4 16 - 4y\,dy \\ &= \left[16y - 2y^2\right]_0^4 \\ &= 64 - 32 \\ &= 32. \end{aligned}$$

Example 7.2.6. Set up the integral representing the volume of a torus with major radius R and minor radius r. Assume $R > r$.



(a) Cross-sectional Area



(b) Revolved Solid

Once again, the horizontal cross-section is a washer with outer radius $R + \sqrt{r^2 - y^2}$ and inner radius $R - \sqrt{r^2 - y^2}$, so the cross-sectional area is

$$A(y) = \pi \left(R + \sqrt{r^2 - y^2}\right)^2 - \pi \left(R - \sqrt{r^2 - y^2}\right)^2$$

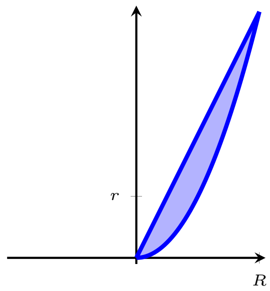
Notice that we'll integrate from $y = -r$ to r. Thus the volume of this torus is given by

$$\begin{aligned} V &= \int_{-r}^{r} A(y)\, dy \\ &= \int_{-r}^{r} \pi \left(R + \sqrt{r^2 - y^2}\right)^2 - \pi \left(R - \sqrt{r^2 - y^2}\right)^2 \, dy. \end{aligned}$$

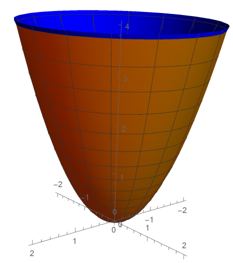
7.3 Volumes by Cylindrical Shells

If we slice our solid of revolution parallel to the axis of rotation, we end up with tiny cylinders of infinitesimal thickness.

Example 7.3.1. Find the volume of the solid generated by rotating the region bounded by $y = 2x$ and $y = x^2$ about the y-axis.



(a) Cross-sectional Area



(b) Revolved Solid

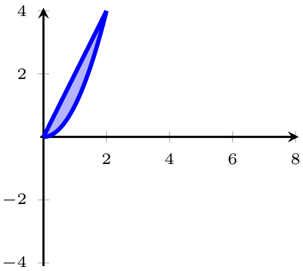
To slice parallel will involve integrating with respect to x. The cylinder will have area $2\pi rh$, so at a specific point x, we get that $r = x$ and the height is the difference of the functions $h = 2x - x^2$, hence at each x, our area is $A(x) = 2\pi x(2x - x^2)$. Our limits of integration are

$$\begin{aligned} 2x &= x^2 \\ x^2 - 2x &= 0 \\ x(x-2) &= 0 \\ \Rightarrow \quad x &= 0, 2. \end{aligned}$$

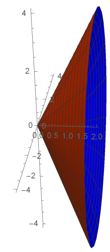
Thus, we integrate over all of these areas and get

$$\begin{aligned} V = \int_0^2 A(x)\,dx &= \int_0^2 2\pi x(2x - x^2)\,dx \\ &= 2\pi \int_0^2 2x^2 - x^3\,dx \\ &= 2\pi \left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2 \\ &= \frac{8\pi}{3}. \end{aligned}$$

Example 7.3.2. Find the volume of the solid generated by rotating the region bounded by $y = 2x$, $y = x^2$ about the x-axis. (Use the method of cylindrical shells)



(a) Cross-sectional Area



(b) Revolved Solid

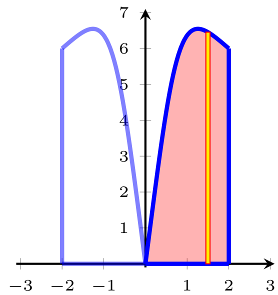
Since cylindrical shells are parallel to the x-axis, we'll be integrating with respect to y. Our functions, rewritten in terms of y, are

$$y = 2x \Rightarrow x = \frac{1}{2}y$$
$$y = x^2 \Rightarrow x = \sqrt{y}$$

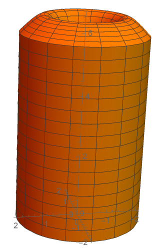
At each fixed value y, our cylindrical shell will have radius y and height $\sqrt{y} - \frac{1}{2}y$. So, the shell's area is $A(y) = 2\pi y\left(\sqrt{y} - \frac{1}{2}y\right)$. The limits of integration will be from 0 to 4. Thus, the volume of this solid is

$$\begin{aligned} V = \int_0^4 A(y)\,dy &= \int_0^4 2\pi y\left(\sqrt{y} - \frac{1}{2}y\right)\,dy \\ &= 2\pi\int_0^4 y^{3/2} - \frac{1}{2}y^2\,dy \\ &= 2\pi\left[\frac{2}{5}y^{5/2} - \frac{1}{6}y^3\right]_0^4 \\ &= \frac{64\pi}{15}. \end{aligned}$$

Example 7.3.3. Using cylindrical shells, find the volume of the region bounded by $y = \frac{9x}{\sqrt{1+x^3}}$, $x = 0$, and $x = 2$, rotated about the y-axis.



(a) Cross-sectional Area



(b) Revolved Solid

Notice that if we try to use a washer, we'll have to use two separate integrals. So instead we use the shell method, integrating with respect to x. Our radius will thus be x and our height will be $\frac{9x}{\sqrt{1+x^3}}$. Our limits of integration are 0 to 2. Thus, the area of each cylindrical shell will be $A(x) = 2\pi x\left(\frac{9x}{\sqrt{1+x^3}}\right)$. Thus, the volume is

$$\begin{aligned} V = \int_0^2 A(x)\,dx &= \int_0^2 2\pi x\left(\frac{9x}{\sqrt{1+x^3}}\right)dx \\ &= 18\pi\int_0^2 \frac{x^2}{\sqrt{1+x^3}}\,dx. \end{aligned}$$

Using the substituion

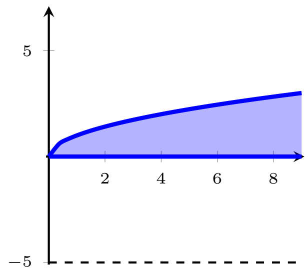
$$\begin{aligned} u &= 1+x^3, & du &= 3x^2dx, \\ u(0) &= 1, & u(2) &= 9, \end{aligned}$$

we get

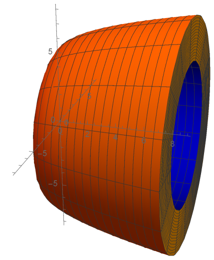
$$\begin{aligned} 18\pi\int_0^2 \frac{x^2}{\sqrt{1+x^3}}\,dx &= 6\pi\int_1^9 \frac{du}{\sqrt{u}} \\ &= 6\pi\left[2\sqrt{u}\right]_1^9 \\ &= 24\pi. \end{aligned}$$

Remark. Beyond simplifying our lives a bit by only requiring one integral, it turns out that there is no way (with only elementary functions) to solve for x in $y = \frac{9x}{\sqrt{1+x^3}}$, so using the disk/washer method is effectively impossible.

Example 7.3.4. Using cylindrical shells, find the volume of the solid generated by rotating the region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$, about the line $y = -5$.



(a) Cross-sectional Area

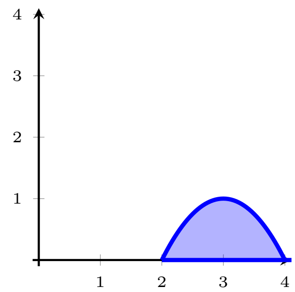


(b) Revolved Solid

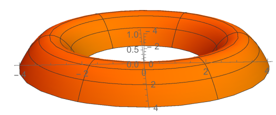
Using cylindrical shells, We see that our limits of integration will be 0 to 3. The radius will be $y - (-5) = y + 5$, and the height of these cylinders will be $9 - y^2$. Thus the area of each cylinder will be $A(y) = 2\pi(y+5)(9-y^2)$. Hence, our volume is

$$\begin{aligned} V = \int_0^3 A(y)\,dy &= \int_0^3 2\pi(y+5)(9-y^2)\,dy \\ &= 2\pi \int_0^3 (-y^3 - 5y^2 + 9y + 45)\,dy \\ &= 2\pi \left[-\frac{1}{4}y^4 - \frac{5}{3}y^3 + \frac{9}{2}y^2 + 45y \right]_0^3 \\ &= \frac{441\pi}{2}. \end{aligned}$$

Example 7.3.5. Set up the definite integral for the volume of the solid generated by rotating the region bounded by $y = -x^2 + 6x - 8$, $y = 0$, about the y-axis.



(a) Cross-sectional Area

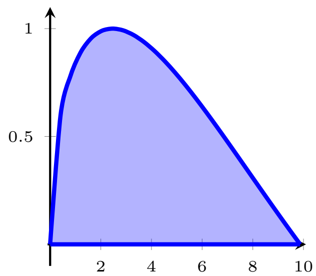


(b) Revolved Solid

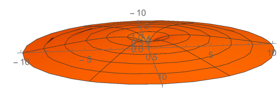
From the graph, we see that the shell method is easier. From the y-axis, the radius is x and the height is $x^2 + 6x - 8$, so the area is given by $A(x) = 2\pi x(x^2 + 6x - 8) = 2\pi(x^3 + 6x^2 - 8x)$. Hence our volume is

$$
\begin{aligned}
V = \int_2^4 A(x)\,dx &= 2\pi \int_2^4 x^3 + 6x^2 - 8x\,dx \\
&= 2\pi \left[\frac{1}{4}x^4 + 2x^3 - 4x^2\right]_2^4 \\
&= 248\pi.
\end{aligned}
$$

Example 7.3.6. Set up the definite integral for the volume of the solid generated by rotating the region bounded by $y = \sin(\sqrt{x})$, $y = 0$, $x = 0$, $x = \pi^2$, about the y-axis.



(a) Cross-sectional Area

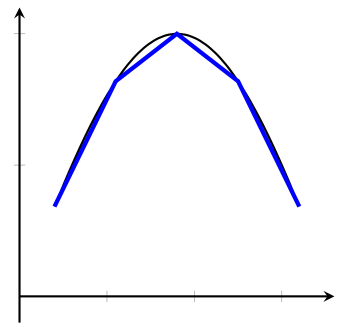

(b) Revolved Solid

From the graph, we see that the shell method is much easier. From the y-axis, the radius is x and the height is $\sin(\sqrt{x})$, so the area is $A(x) = 2\pi x \sin(\sqrt{x})$. Thus the volume is

$$\begin{aligned} V = \int_0^{\pi^2} A(x)\, dx &= 2\pi \int_0^{\pi^2} x \sin(\sqrt{x})\, dx \\ &= 4\pi \int_{u=0}^{u=\pi} u \sin(u)\, du && \text{(substitution } u = x^{1/2},\ du = \tfrac{1}{2}x^{-1/2}\, dx\text{)} \\ &= 4\pi \left[\sin(u) - u\cos(x)\right]_0^{\pi} && \text{(integrating by parts)} \\ &= 4\pi^2. \end{aligned}$$

7.4 Arc Length

Approximate a smooth curve $y = f(x)$ with a bunch of line segments, like so:



Each of these line segments has length $L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. So we have

$$\begin{aligned} L &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \\ &= \sqrt{(\Delta x)^2 + \left(\frac{\Delta y}{\Delta x}\right)^2 (\Delta x)^2} \\ &= \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \cdot \Delta x. \end{aligned}$$

The length of the arc over the interval $[a, b]$ is just approximately the sum of each of these lengths. As the number of segments increases, our accuracy increases and the length Δx becomes the infinitesimally tiny dx. As well, $\frac{\Delta y}{\Delta x}$ becomes $\frac{dy}{dx} = f'(x)$. This gives us the following useful fact:

Fact. The **arc length** of the curve $y = f(x)$ on the interval $[a, b]$ is given by

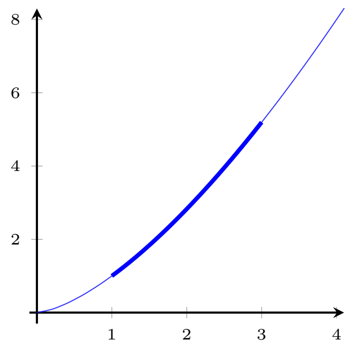
$$L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Similarly, the **arc length** of the curve $x = g(y)$ on the interval $[h, k]$ is given by

$$L = \int_h^k \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy = \int_h^k \sqrt{1 + [g'(y)]^2}\, dy.$$

Remark. This is not the mathematical definition of arc length, but rather a very specific case when your arc is of the form $y = f(x)$ or $x = g(y)$. See your book or an introductory differential geometry text for more discussion on arc length.

Example 7.4.1. Find the arc length of the curve $y = x^{3/2}$ for $1 \leq x \leq 3$.



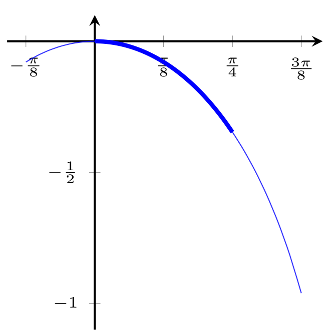
We'll first work with the radicand.

$$\begin{aligned} 1 + (y')^2 &= 1 + \left(\frac{3}{2}x^{1/2}\right)^2 \\ &= 1 + \frac{9}{4}x \end{aligned}$$

So, the arc length is

$$\begin{aligned} L &= \int_1^3 \sqrt{1 + \frac{9}{4}x}\, dx \\ &= \frac{4}{9}\int_{13/4}^{31/4} \sqrt{u}\, du \\ &= \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_{13/4}^{31/4} \\ &= \frac{1}{27}\left(31\sqrt{31} - 13\sqrt{13}\right). \end{aligned}$$

Example 7.4.2. Find the arc length of the curve $y = \ln(\cos(x))$ for $0 \leq x \leq \frac{\pi}{4}$.



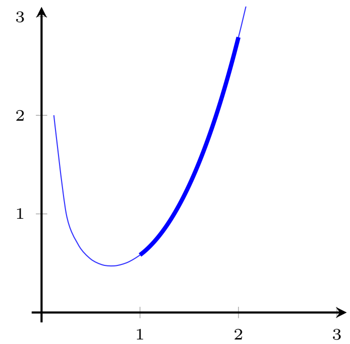
We'll first work with the radicand.

$$\begin{aligned} 1 + (y')^2 &= 1 + \tan^2(x) \\ &= \sec^2 x. \end{aligned}$$

So, the arc length is

$$\begin{aligned} L &= \int_0^{\pi/4} \sqrt{\sec^2(x)}\, dx \\ &= \int_0^{\pi/4} \sec(x) \\ &= \left[\ln|\sec(x) + \tan(x)|\right]_0^{\pi/4} \\ &= \ln(\sqrt{2} + 1) - \ln(1) \approx 0.881. \end{aligned}$$

Example 7.4.3. Find the arc length of the curve $y = \frac{1}{3}x^3 + \frac{1}{4x}$ for $1 \leq x \leq 2$.



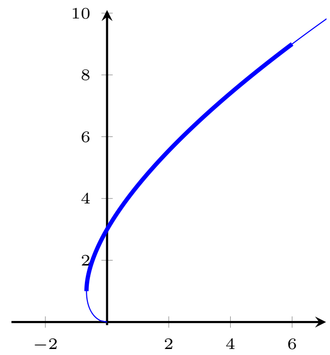
Again we work with the radicand first.

$$\begin{aligned} 1 + (y')^2 &= 1 + \left(x^2 - \frac{1}{4x^2}\right)^2 \\ &= 1 + \left(x^4 - \frac{1}{2} + \frac{1}{16x^4}\right) \\ &= x^4 + \frac{1}{2} + \frac{1}{16x^4} \\ &= \left(x^2 + \frac{1}{4x^2}\right)^2 \end{aligned}$$

So, our arc length becomes

$$\begin{aligned} L &= \int_1^2 \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2}\, dx \\ &= \int_1^2 x^2 + \frac{1}{4x^2}\, dx \\ &= \left[\frac{1}{3}x^3 - \frac{1}{4x}\right]_1^2 \\ &= \frac{59}{24}. \end{aligned}$$

Example 7.4.4. Find the arc length of the curve $x = \frac{1}{3}\sqrt{y}(y-3)$, from $y = 1$ to $y = 9$.



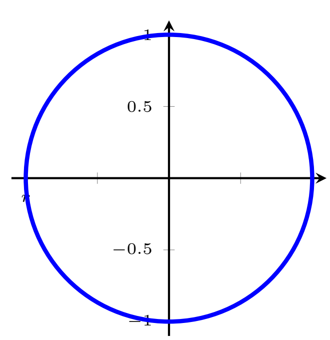
Again, we start with just the radicand.

$$\begin{aligned}
1 + (x')^2 &= 1 + \left(\frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}\right)^2 \\
&= 1 + \left(\frac{1}{4}y - \frac{1}{2} + \frac{1}{4}y^{-1}\right) \\
&= \frac{1}{4}y + \frac{1}{2} + \frac{1}{4}y^{-1} \\
&= \left(\frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\right)^2
\end{aligned}$$

Our arc length is then

$$\begin{aligned}
L &= \int_1^9 \sqrt{\left(\frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\right)^2}\,dy \\
&= \int_1^9 \frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\,dy \\
&= \left[\frac{1}{3}y^{3/2} + y^{1/2}\right]_1^9 \\
&= \frac{32}{3}
\end{aligned}$$

Example 7.4.5. Find the circumference of a circle of radius r.



This is left as a challenge to the reader, who may recall that $x^2 + y^2 = r^2$. (Hint: Try computing the arc length of just one quadrant of the circle.)

7.6 Applications to Physics and Engineering

Recall that the force equation is $F = ma = m\frac{d^2s}{dt^2} = \rho g V$ (where ρ is density and g is gravity's acceleration) and that the work equation is $W = Fd$, where d is distance. When the force is constant, it's easy to find the work done, but what if the force is not constant? Well, we can approximate the work done by breaking up our distance d into smaller subintervals, treating the work as constant on each subinterval, and then summing those together. To get an exact answer, we take a limit over more and more subintervals. Well, this is exactly a procedure of Riemann sums, so we get the following

Definition. The work done in moving an object from a to b is

$$W = \int_a^b F(x)\,dx,$$

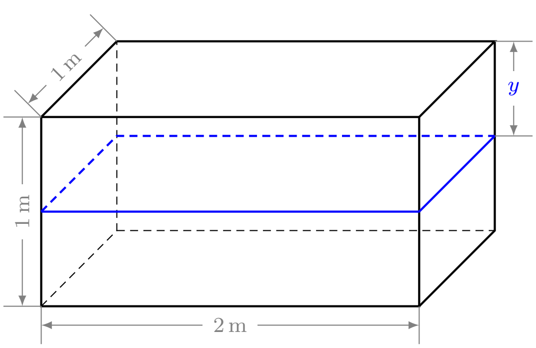
where $F(x)$ is the variable force of the object.

In SI units work is in measured in Joules (denoted J), and in imperial units work is measured in foot-pounds (denoted ft-lb).

Example 7.6.1. A variable force of $f(x) = 5x^{-2}$ lb moves an object along a straight line when the object is x ft from the origin. Calculate the work done moving the object from $x = 1$ ft to $x = 10$ ft.

$$W = \int_1^{10} f(x)\,dx = \int_1^{10} 5x^{-2}\,dx = \left[-5x^{-1}\right]_1^{10} = \frac{9}{2}\text{ft-lb}.$$

Example 7.6.2. An aquarium 2 m long, 1 m wide, and 1 m deep is full of water. How much work is done when pumping all of the water out of the top of the tank? How much work is done when pumping only half of the water out?



Each "slice" of water is dy thick, and has volume $2dy\,\mathrm{m}^3$. Recall that, in SI units, water weighs $9800\,\mathrm{kg/m}^3$, and so the incremental force function is $9800 \cdot 2dy = 19600dy$. Moving each "slice" of water up y units to the top of the tank until the tank is empty gives us the following:

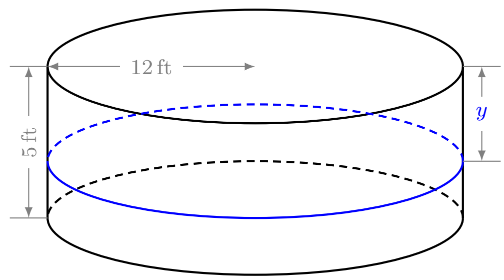
$$W = \int_0^1 9800(y)(2)\,dy = 19600\int_0^1 y\,dy = 19600\left[\frac{1}{2}y^2\right]_0^1 = 9800\,\mathrm{J}.$$

To only empty half of the tank, notice that y varies from 0 to 1/2, so we get that the amount of work done is

$$W = 19600\int_0^{1/2} y\,dy = 19600\left[\frac{1}{2}y^2\right]_0^{1/2} = 2450\,\mathrm{J}.$$

Remark. When non-constant forces are involved, moving half the distance is not equivalent to doing only half the work.

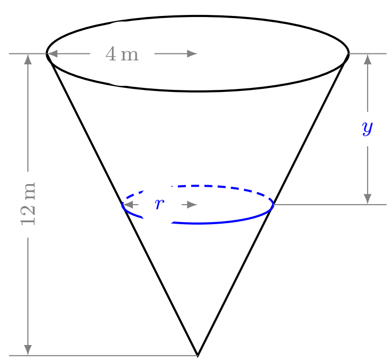
Example 7.6.3. A circular swimming pool has a diameter of 24 ft, sides of 5 ft in height, and a water level of 4 ft. How much work is required to pump all of the water out of the top of pool?



Each "slice" of water is dy thick, and has volume $\pi(12)^2 dy = 144\pi dy\,\text{ft}^3$. Recall that water weighs $62.5\,\text{lb/ft}^3$, and so the incremental force function is given by $62.5(144\pi)dy = 9000\pi dy$. Note that y starts at 1 because the water level is 1 ft below the top of the pool. Moving each "slice" of water up y units to the top of the tank until the tank is empty gives us the following:

$$W = \int_1^5 9000\pi y\,dy = 9000\pi \int_1^5 y\,dy = 9000\pi \left[\frac{1}{2}y^2\right]_1^5 = 108,000\pi\ \text{ft-lb}.$$

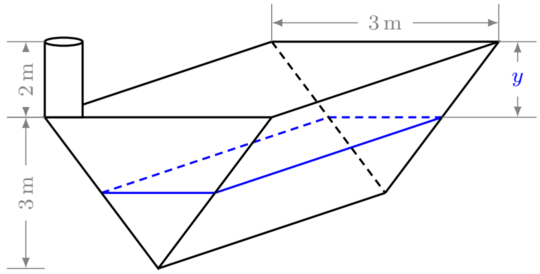
Example 7.6.4. An tank shaped like an inverted right-angled circular cone, with height 12 m and base radius 4 m, is full of water. Water is pumped out through the top of the tank with a hose until the water level is 4 m high. How much work is done in pumping the water out?



By similar triangles, the "slice" of water shown in the diagram has radius $r = 4(12-y)/12$. So, given a thickness of dy, each "slice" has volume $\pi r^2 dy = (\pi/9)(12-y)^2 dy$. Recall that, in SI units, water weighs $9800\,\text{kg/m}^3$, and so the incremental force function is $(9800\pi/9)(12-y)^2 dy$. Note that y stops at 8 because we want to leave 4 m of water in the tank. Moving each "slice" of water up y units to the top of the tank until the tank has only 4 m of water gives us the following

$$W = \frac{9800\pi}{9}\int_0^8 (12-y)^2 y\,dy = \frac{9800\pi}{9}\left[\frac{1}{4}y^4 - 8y^3 + 72y^2\right]_0^8 = \frac{5,017,600\pi}{3}\ \text{J} \approx 5.2544\times 10^6\ \text{J}.$$

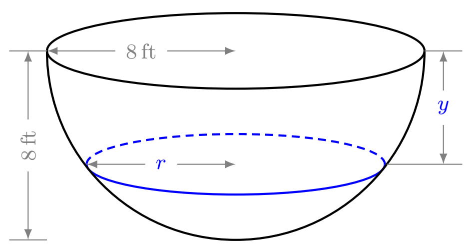
Example 7.6.5. A triangular trough 3 m high, 3 m wide, and 8 m long is full of water. Water is pumped out of the top of the trough through a 2 m tall spigot. How much work is required to empty the trough?



By similar triangles, the width w of the water "slice" is $w = 3 - y$, and since each slice is dy thick and 8 m long, we have that the volume of each slice is given by $8(3 - y)dy$. Recall that, in SI units, water weighs $9800\,\mathrm{kg/m^3}$, and so the incremental force function is $9800(8)(3 - y)dy$. Moving each "slice" up $y + 2$ meters (y for the tank, 2 for the spigot) until the tank is empty, we have that the work is given by

$$W = 78400 \int_0^3 (3 - y)(y + 2)\,dy = 78400 \left[-\frac{1}{3}y^3 + \frac{1}{2}y^2 + 6y\right]_0^3 = 1,058,400\,\mathrm{J}.$$

Example 7.6.6. A hemispherical tank with radius 8 ft is filled with water. Find the amount of work done in pumping all of the water out of the tank.



We can form a right triangle with height y, base r, and hypotenuse 8. At height y, applying the Pythagorean theorem yields that the radius of each "slice" of water is $r = \sqrt{64 - y^2}$. Each "slice" of water is dy thick, and has volume $\pi(64 - y^2)dy\,\mathrm{ft}^3$. Recall that water weighs $62.5\,\mathrm{lb/ft^3}$, and so the incremental force function is given by $62.5\pi(64 - y^2)dy$. Moving each "slice" of water up y units to the top of the tank until the tank is empty gives us the following:

$$W = \int_0^8 62.5\pi(64 - y^2)\,dy = 62.5\pi\left[64y - \frac{1}{3}y^3\right]_0^8 = \frac{64,000\pi}{3}\,\text{ft-lb} \approx 67020.6\,\text{ft-lb}.$$

Example 7.6.7. A 15 ft chain weighing 3 lb per foot is lying coiled on the ground. How much work is required to raise one end of the chain to a height of 15 ft?

Notice that the force changes with every foot of chain lifted. So at height y, there are y ft of chain off the ground, which means our force function is $F(y) = 3y$ lb. And so, the total work required is

$$W = \int_0^{15} 3y\,dy = \left[\frac{3}{2}y^2\right]_0^{15} = \frac{675}{2}\text{ ft-lb.}$$

Example 7.6.8. A force of 100 lb is required to compress a spring 6 in from its natural length of 2 ft. Find the work done (in ft-lb) in compressing the spring an additional 3 in.

First we'll do some conversions: 6 in = 0.5 ft and 3 in = 0.25 ft. Now using Hooke's law, $F(x) = kx$, we first have to solve for the spring constant k.

$$100 = k(0.5) \quad \Rightarrow k = 200,$$

and thus the force function for this spring is given by $F(x) = 200x$. So, to compress it an *additional* 3 in, we need to integrate from 6 in to 9 in:

$$W = \int_{1/2}^{3/4} 200x\,dx = \left[100x^2\right]_{1/2}^{3/4} = \frac{125}{4} = 31.25\text{ ft-lb.}$$

8 Series

8.1 Sequences

Definition. A **sequence** of real numbers is the image of a function $a : \mathbb{N} \to \mathbb{R}$, but can be thought of as an ordered list of real numbers

$$\{a_1, a_2, a_3, \ldots\},$$

We often denote the sequence as $\{a_n\}_{n=1}^{\infty}$ or just $\{a_n\}$ (some authors replace the curly braces with parentheses instead).

Frequently, you will see sequences written in one of three different ways: using the aforementioned notation, giving a defining function for a_n, or explicitly writing out the terms of the sequence. We also note that n does not necessarily have to start at 1.

Example 8.1.1. The following sequential descriptions are equivalent:

$$\left\{\frac{1}{n^2}\right\}_{n=3}^{\infty} \qquad a_n = \frac{1}{n^2},\ n \geq 3 \qquad \left\{\frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}, \ldots, \frac{1}{n^2}, \ldots\right\}.$$

For the most part, it's useful to find an explicit description for each n^{th} term, as in the first or second way of writing it.

Example 8.1.2. Given the sequence $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$, find an explicit description for a_n.

Notice that the numerator and denominator both increase by 1 each time, and that the denominator is always 1 more than the numerator. Thus, we can write

$$a_n = \frac{n}{n+1},\ n \geq 1 \quad \text{or} \quad \left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$$

Example 8.1.3. Given the sequence $\left\{\frac{-1}{2}, \frac{2}{3}, \frac{-3}{4}, \frac{4}{5}, \ldots\right\}$, find an explicit description for a_n.

Notice this is the same sequence as in Example 8.1.2, except when n is odd, we have that the term is negative, and when n is even, we have that the term is positive. Thus, we can use the fact that $(-1)^n$ is negative for n is odd and positive for when n is even, which gives us

$$a_n = \frac{(-1)^n n}{n+1},\ n \geq 1 \quad \text{or} \quad \left\{\frac{(-1)^n n}{n+1}\right\}_{n=1}^{\infty}.$$

Since sequences are just discrete functions, it may be useful to see what sorts of things we can do with them. Much like we've seen before, we can discuss limits of sequences.

Definition. A sequence $\{a_n\}$ is **convergent** if there exists a real number L so that

$$\lim_{n\to\infty} a_n = L.$$

If a sequence is not convergent, then we say that the sequence is **divergent**.

The following theorem tells us that we can deal with limits of sequences by using many of our previous techniques for functions on the real numbers:

Theorem 8.1.4. *If $f : \mathbb{R} \to \mathbb{R}$ with $f(n) = a_n$ (where n is an integer) and $\lim_{x\to\infty} f(x) = L$, then*

$$\lim_{n\to\infty} a_n = L.$$

As a result of this limit correspondence, we have the following statements about convergent sequences:

Proposition 8.1.5. *Let $\{a_n\}$, $\{b_n\}$ be convergent sequences and c a constant. Then,*

1. $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty} c \cdot a_n = c \cdot \left(\lim_{n\to\infty} a_n\right)$
3. $\lim_{n\to\infty} (a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\pm \lim_{n\to\infty} b_n\right)$
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ *if* $\lim_{n\to\infty} b_n \neq 0$
5. $\lim_{n\to\infty} (a_n)^p = \left(\lim_{n\to\infty} a_n\right)^p$ *if $p > 0$ and $a_n > 0$*

With these rules, we can prove the following useful fact

Proposition 8.1.6. *If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.*

The Squeeze Theorem can also be adapted for sequences

Theorem 8.1.7 (Squeeze)**.** *If $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.*

Example 8.1.8. Find the limit of the sequence, if it converges. $\left\{\frac{37n+16}{9n-42}\right\}_{n=5}^{\infty}$.

We can appeal to the limit laws in Proposition 8.1.5:

$$\begin{aligned}
\lim_{n\to\infty} \frac{37n+16}{9n-42} &= \lim_{n\to\infty} \frac{n\left(37+\frac{16}{n}\right)}{n\left(9-\frac{42}{n}\right)} \\
&= \lim_{n\to\infty} \frac{\left(37+\frac{16}{n}\right)}{\left(9-\frac{42}{n}\right)} \\
&= \frac{\lim_{n\to\infty}\left(37+\frac{16}{n}\right)}{\lim_{n\to\infty}\left(9-\frac{42}{n}\right)} \\
&= \frac{\lim_{n\to\infty} 37 + \lim_{n\to\infty} \frac{16}{n}}{\lim_{n\to\infty} 9 - \lim_{n\to\infty} \frac{42}{n}} \\
&= \frac{37}{9}.
\end{aligned}$$

Example 8.1.9. Find the limit of the sequence, if it converges. $\left\{\frac{(-1)^n}{n^2}\right\}_{n=1}^{\infty}$.

Since $\left|\frac{(-1)^n}{n^2}\right| = \frac{1}{n^2}$ and $\lim_{n\to\infty} \frac{1}{n^2} = 0$, then the given sequence also converges to 0.

Example 8.1.10. Find the limit of the sequence, if it converges. $\{2\arctan(n)\}_{n=0}^{\infty}$.

Since $f(x) = 2\arctan(x)$ agrees with a_n for each n, we have that

$$\lim_{n\to\infty} 2\arctan(n) = \lim_{n\to\infty} 2\arctan(x) = \pi.$$

The sequences we've seen so far are defined explicitly. Sometimes, however, we may see sequences defined recursively.

Example 8.1.11. The well-known **Fibonacci sequence** $\{F_n\}$ is defined recursively by

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3.$$

This gives us

$$\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}.$$

Example 8.1.12. Find an explicit formula for each term in the given recursive sequence:

$$a_0 = 1, \quad a_n = 2a_n \text{ for } n \geq 1.$$

Writing out the sequence, we have

$$\{1, 2, 4, 8, 16, 32, 64, 128, \ldots\},$$

which we recognize as the sequence where $a_n = 2^n$.

8.2 Series

Definition. The sum of all of the terms in a sequence $\{a_n\}_{n=1}^{\infty}$ is called a **series**, and is denoted $\displaystyle\sum_{n=1}^{\infty} a_n$ or $\sum a_n$.

How do we make sense of talking about the value of an infinite series? As usual, we'll use limits.

Definition. The **partial sums** for the series $\displaystyle\sum_{n=1}^{\infty} a_n$ are

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_k &= \sum_{n=1}^{k} a_n. \end{aligned}$$

For lack of better terminology on the author's part, we'll call s_k the $\boldsymbol{k}^{\mathbf{th}}$ **partial sum**, as it is the sum of all terms up to (and including) the k^{th} term.

Note: if the series starts at some number $i \neq 1$ then we still take

$$s_k = a_i + \cdots + a_{k-1} + a_{k-1} + a_k = \sum_{n=i}^{k} a_n$$

for the purposes of notational simplicity. The common definition in the literature is to define the k^{th} partial sum as the sum of the first k terms. We will write "the sum of the first k terms" whenever necessary to avoid ambiguity.

Definition. Let s_k denote the k^{th} partial sum of this series $\displaystyle\sum_{n=1}^{\infty} a_n$. We say that this series is **convergent** if there exists a real number s so that

$$\lim_{k\to\infty} s_k = s.$$

If this series converges, we write

$$\sum_{n=1}^{\infty} a_n = s.$$

We call s the **sum of the series**. If the sequence of partial sums $\{s_k\}$ does not converge, we say that the series is **divergent**.

Example 8.2.1. A **geometric series** is a series of the form

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{n=0}^{\infty} ar^n,$$

where $a \neq 0$ is some constant and r is called the **common ratio**. The variable a doesn't have a name, but it is useful to think of it as the first term in a geometric series.

For a geometric series, we can obtain a nifty formula for the partial sum s_k:

$$\begin{aligned} s_k &= a + ar + ar^2 + \cdots + ar^k \\ rs_k &= ar + ar^2 + ar^3 + \cdots + ar^{k+1} \end{aligned}$$

Then

$$\begin{aligned} s_k - rs_k &= a - ar^{k+1} \\ s_k(1-r) &= a(1-r^{k+1}) \\ \Rightarrow s_k &= \sum_{n=0}^{k} ar^n = \frac{a(1-r^{k+1})}{1-r}. \end{aligned}$$

If $|r| < 1$, then $\lim\limits_{k\to\infty} |r|^k = 0$, so $\lim_{k\to\infty} r = 0$ by Proposition 8.1.6, and it is easy to see that, for the partial sums above,

$$\lim_{k\to\infty} s_k = \frac{a}{1-r}.$$

We can also see that the series diverges when $|r| > 1$, and the case when $|r| = 1$ is handled in your book. These combined lead us to the following result

Theorem 8.2.2 (Geometric Series Test)**.** *If $|r| < 1$,*

$$\sum_{n=0}^{\infty} ar^n \textit{ converges,} \quad \textit{and} \quad \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

If $|r| \geq 1$,

$$\sum_{n=0}^{\infty} ar^n \textit{ diverges.}$$

Example 8.2.3. Find the value of the geometric series where $a = 5$ and $r = \frac{1}{2}$, if it converges. If the series diverges, clearly state that it diverges.

The series converges since $|r| = \left|\frac{1}{2}\right| = \frac{1}{2} < 1$. Thus, by the previous theorem, we have

$$\sum_{n=0}^{\infty} 5\left(\frac{1}{2}\right)^n = \frac{5}{1-\frac{1}{2}} = 10.$$

Example 8.2.4. Find the value of the geometric series $\displaystyle\sum_{k=1}^{\infty} \frac{\pi^{k+1}}{e^k}$, if it converges.

First we need to find the common ratio. Notice that we can rewrite the series slightly as

$$\sum_{k=1}^{\infty} \frac{\pi \cdot \pi^k}{e^k} = \sum_{k=1}^{\infty} \pi \left(\frac{\pi}{e}\right)^k.$$

In this form, we have that $r = \frac{\pi}{e}$. Since $|r| > 1$, then the series diverges.

Example 8.2.5. Find the value of the geometric series $\sum_{n=5}^{\infty} 10\left(\frac{1}{2}\right)^n$, if it converges.

We see that $r = \frac{1}{2}$, and so by the geometric series test, this series converges. Although we'd like to use the formula in the geometric series test to determine the value, we have to be careful because that series starts at $n = 0$ and ours starts at $n = 5$. Recalling that a was the first term in our series, this means that our first term is $a = 10\left(\frac{1}{2}\right)^5 = \frac{5}{16}$, and so we have that

$$\sum_{n=5}^{\infty} 10\left(\frac{1}{2}\right)^n = \frac{\frac{5}{16}}{1-\frac{1}{2}} = \frac{\frac{5}{16}}{\frac{1}{2}} = \frac{5}{8}.$$

An alternative method to solving this problem is to do what's called an "index shift" to make the series start at 0. Begin by letting $m = n - 5$. Then when $n = 5$, $m = 0$, and as $n \to \infty$, $m \to \infty$ as well. So we get

$$\sum_{n=5}^{\infty} 10\left(\frac{1}{2}\right)^n = \sum_{m=0}^{\infty} 10\left(\frac{1}{2}\right)^{m+5} = \sum_{m=0}^{\infty} 10\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^m = \sum_{m=0}^{\infty} \frac{5}{16}\left(\frac{1}{2}\right)^m,$$

and this means that $a = \frac{5}{16}$ again, so we get the same sum with this index shifting approach as well.

One reason we like convergent series so much is because the following result

Proposition 8.2.6. *If $\sum a_n$, $\sum b_n$ are convergent series and c is a constant, then $\sum ca_n$ and $\sum(a_n \pm b_n)$ are convergent series and*

1. $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$,
2. $\displaystyle\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$.

Example 8.2.7. Find the sum of the following series, if it converges. $\displaystyle\sum_{m=1}^{\infty} \frac{6+6^m}{8^m}$.

We notice that

$$\sum_{m=1}^{\infty} \frac{6}{8^m} = \frac{\frac{6}{8}}{1-\frac{1}{8}} = \frac{6}{7}$$

and

$$\sum_{m=1}^{\infty} \frac{6^m}{8^m} = \frac{\frac{6}{8}}{1-\frac{6}{8}} = 3,$$

by since both converge, by Proposition 8.2.6, we have

$$\sum_{m=1}^{\infty} \frac{6+6^m}{8^m} = \sum_{m=1}^{\infty} \frac{6}{8^m} + \sum_{m=1}^{\infty} \frac{6^m}{8^m} = \frac{6}{7} + 3 = \frac{27}{7}.$$

Example 8.2.8 (Telescoping Series). Find the sum of the following series, if it converges. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$

This sequence is not obviously a geometric series, so we'll have to approach by a sequence of partial sums. Notice, however, that we can apply partial fractions to the summand:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2} &= \sum_{n=1}^{\infty} \frac{1}{(n+2)(n+1)} \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)\end{aligned}$$

And so, each k^{th} partial sum can be written

$$s_k = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{k+1} - \frac{1}{k+2}\right) = \frac{1}{2} - \frac{1}{k+2},$$

and so

$$\sum_{n=1}^{\infty} \frac{1}{n^2+3n+1} = \lim_{k\to\infty} s_k = \lim_{k\to\infty} \frac{1}{2} - \frac{1}{k+2} = \frac{1}{2} + 0 = \frac{1}{2}.$$

Up to this point, we've been able to explicitly calculate the sum for these various series. In general, the best we can hope to do is to show that the series converges at all (at which point we can use a computer to approximate the value numerically).

Example 8.2.9 (Harmonic Series). Determine whether or not the following series converges. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

As a heuristic, we expect this to behave similarly to $\int_1^{\infty} \frac{1}{x}\, dx$, and thus suspect it might converge. To see that we are correct, we need to consider the limit of partial sums. But in fact, we can restrict our attention to only some of the partial sums: s_2, s_4, s_8, s_{16}, etc.

$$\begin{aligned} s_2 &= 1 + \frac{1}{2} \\ s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2} \\ s_8 &= \cdots > 1 + \frac{3}{2} \\ s_{16} &= \cdots > 1 + \frac{4}{2} \\ &\vdots \\ s_{2^n} &= \cdots > 1 + \frac{n}{2} \end{aligned}$$

So, as $n \to \infty$, we see also that $s_{2^n} \to \infty$, and thus the series above (called the **harmonic series**) diverges.

Theorem 8.2.10 (Divergence Test)**.** *If* $\lim_{n\to\infty} a_n \neq 0$*, then the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Note the order of the logic in this theorem. The converse is *not* true, and the harmonic series provides the counter example. Also note that the name of the test tells us exactly what its limitations are – *it tests for divergence; not convergence!*

Example 8.2.11. Determine whether or not the following series converges. $\sum_{n=0}^{\infty} \frac{7^n}{3^{(n+41)}}$

Although we recognize that this is a geometric series, we can also see that

$$\lim_{n\to\infty} \frac{7^n}{3^{(n+41)}} = \infty,$$

and so by the previous theorem (8.2.10), the whole series diverges.

8.3 The Integral and Comparison Tests

Thinking of the series $\sum a_n$ as the left-Riemann sums of a function f, we get the following test

Theorem 8.3.1 (Integral Test)**.** *Suppose f is a continuous, positive, decreasing function on $[k, \infty)$ and let $a_n = f(n)$.*

1. *If $\displaystyle\int_k^\infty f(x)\,dx$ converges, then $\displaystyle\sum_{n=k}^\infty a_n$ converges.*
2. *If $\displaystyle\int_k^\infty f(x)\,dx$ diverges, then $\displaystyle\sum_{n=k}^\infty a_n$ diverges.*

Example 8.3.2. Determine if the series $\displaystyle\sum_{n=2}^\infty \frac{1}{n \ln n}$ is convergent or divergent.

Use the function $f(x) = \frac{1}{x \ln x}$. For x in the interval $[2, \infty)$, this function is positive. As well, as x increases, the denominator increases, and so the output decreases. So, we determine convergence of the following integral:

$$\begin{aligned}
\int_2^\infty \frac{1}{x \ln x}\,dx &= \lim_{t\to\infty} \int_2^t \frac{1}{x \ln x}\,dx \\
&= \lim_{t\to\infty} [\ln(\ln x)]_2^t && \text{(substitution of } u = \ln x) \\
&= \lim_{t\to\infty} \ln(\ln t) - \ln(\ln 2)) \\
&= \infty.
\end{aligned}$$

So since the integral diverges, the series diverges.

Now that we have this integral test, combining it with the results of Example 6.6.4 gives us

Theorem 8.3.3 (p-Series Test)**.** *The p-series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.*

Example 8.3.4. Determine if the series $\displaystyle\sum_{n=2}^\infty \frac{1}{\sqrt{n}}$ converges or diverges.

We have that this series is a p-series with $p = \frac{1}{2}$. So by the p-series test, this series diverges.

The following result is also an adaptation of the comparison test for integrals in Section 6.6.

Theorem 8.3.5 (Comparison Test)**.** *Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.*

1. *If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.*
2. *If $\sum b_n$ is divergent and $b_n \leq a_n$ for all n, then $\sum a_n$ is also divergent.*

Example 8.3.6. Determine whether or not the following series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{2n}$

We have that, for all n,

$$\frac{3}{2n} = \frac{3}{2} \cdot \frac{1}{n} > \frac{1}{n}.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, then by statement 2 in the comparison test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{3}{2n}$ diverges also.

Example 8.3.7. Determine whether or not the following series converges or diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3n^2}$

We have that, for all n,

$$\frac{2}{3n^2} = \frac{2}{3} \cdot \frac{1}{n^2} < \frac{1}{n^2}.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, then by statement 1 in the comparison test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3n^2}$ converges also.

8.4 Other Convergence Tests

8.4.1 Alternating Series

Definition. An **alternating series** is a series of the form

$$\sum_{n=1}^{\infty}(-1)^n b_n \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n+1} b_n,$$

where b_n is a nonnegative number.

Example 8.4.1. Summing the terms of the sequence in Example 8.1.3, we have an alternating series.

Example 8.4.2. The following is an alternating series (sometimes called the **alternating harmonic series**):

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}.$$

Theorem 8.4.3 (Alternating Series Test)**.** *If the alternating series*

$$\sum_{n=1}^{\infty}(-1)^n b_n$$

satisfies both

1. $b_n \geq b_{n+1}$ *for all* n*, and*
2. $\lim_{n\to\infty} b_n = 0$*,*

then the series is convergent.

Example 8.4.4. Test the alternating harmonic series for convergence or divergence.

We have

$$b_n = \frac{1}{n} > \frac{1}{n+1} = b_{n+1},$$

so we satisfy the first condition of the alternating series test. Also,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n} = 0,$$

so we satisfy the second condition of the alternating series test. Therefore the alternating harmonic series converges.

Because of alternating series, we can talk about varying strengths of convergence of a series.

Definition. A series $\sum a_n$ is said to be **absolutely convergent** if $\sum |a_n|$ converges.

Definition. A series $\sum a_n$ is said to be **conditionally convergent** if it converges, but not absolutely.

Example 8.4.5. Does the alternating harmonic series converge conditionally or absolutely?

We've seen that the alternating harmonic series converges. However, since

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n},$$

we have that the alternating harmonic series does not converge absolutely. Therefore it converges conditionally.

8.4.2 Ratio Test

The following test is arguably the most useful test (the proof of which can be found in the course text)

Theorem 8.4.6 (Ratio Test)**.**

1. *If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$*, then the series* $\sum a_n$ *is absolutely convergent (hence, convergent).*
2. *If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$*, then the series* $\sum a_n$ *is divergent.*
3. *If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$*, then the test is inconclusive about the convergence or divergence of the series* $\sum a_n$*.*

The ratio test nearly always the go-to when there are exponents involving n and/or factorials.

For purposes of simplifying fractions, it may behoove you to remember that

$$(n+1)! = (n+1)n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = (n+1)\cdot n!.$$

Example 8.4.7. Determine if the following series converges or diverges. $\displaystyle\sum_{n=0}^{\infty}\frac{n!}{2^n}$

Applying the ratio test, we have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{2^{n+1}}}{\frac{n!}{2^n}}\right| \\ &= \lim_{n\to\infty}\left|\frac{(n+1)!}{2^{n+1}}\cdot\frac{2^n}{n!}\right| \\ &= \lim_{n\to\infty}\left|\frac{n+1}{2}\right| > 1.\end{aligned}$$

So, by the ratio test, our series diverges.

Example 8.4.8. Determine if the following series converges or diverges. $\displaystyle\sum_{k=1}^{\infty}\frac{k^3}{(\ln 3)^k}$

Applying the ratio test, we have

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{\frac{(k+1)^3}{(\ln 3)^{k+1}}}{\frac{k^3}{(\ln 3)^k}}\right| \\ &= \lim_{k\to\infty}\left|\frac{(k+1)^3}{(\ln 3)^{k+1}}\cdot\frac{(\ln 3)^k}{k^3}\right| \\ &= \lim_{k\to\infty}\left|\frac{(k+1)^3}{(\ln 3)k^3}\right| = \frac{1}{\ln 3} < 1.\end{aligned}$$

So, by the ratio test, our series converges absolutely (hence the series converges).

Example 8.4.9. Determine if the following series converges or diverges. $\displaystyle\sum_{n=2}^{\infty} \frac{n^2 2^n}{5^n}$

Applying the ratio test, we have

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{(n+1)^2 2^{n+1}}{5^{n+1}}}{\frac{n^2 2^n}{5^n}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)^2 2^{n+1}}{5^{n+1}} \cdot \frac{5^n}{n^2 2^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{2(n+1)^2}{5n^2} \right| = \frac{2}{5} < 1.
\end{aligned}$$

So, by the ratio test, our series converges absolutely (hence the series converges).

Example 8.4.10. Determine if the following series converges or diverges. $\displaystyle\sum_{p=1}^{\infty} \frac{(2p)!}{(p!)^2}$

Applying the ratio test, we have

$$\begin{aligned}
\lim_{p\to\infty} \left| \frac{a_{p+1}}{a_p} \right| &= \lim_{p\to\infty} \left| \frac{\frac{(2p+2)!}{[(p+1)!]^2}}{\frac{(2p)!}{(p!)^2}} \right| \\
&= \lim_{p\to\infty} \left| \frac{(2p+2)!}{[(p+1)!]^2} \cdot \frac{(p!)^2}{(2p)!} \right| \\
&= \lim_{p\to\infty} \left| \frac{(2p+2)(2p+1)}{(p+1)^2} \right| = 4 > 1.
\end{aligned}$$

So, by the ratio test, our series diverges.

8.5 Power Series

Definition. A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are the **coefficients** of the series. More generally, a **power series centered at a** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

We note that a power series should always start at $n = 0$. If the series is written to start at $n = k > 0$, then it is assumed that $c_0 = \cdots = c_{k-1} = 0$.

Given a power series, we can talk about its convergence for each particular x-value.

Example 8.5.1. For which values of x does the series converge? $\displaystyle\sum_{n=0}^{\infty} x^n$

We see that this looks just like a geometric series with first term $a = 1$ and ratio x. By the geometric series test, this series converges for all x where $|x| < 1$.

Example 8.5.2. For which values of x does the series converge? $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-5)^n}{n}$

Applying the Ratio Test, we have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{\frac{(2x-5)^{n+1}}{n+1}}{\frac{(2x-5)^n}{n}}\right| &= \lim_{n\to\infty}\left|\frac{(2x-5)^{n+1}}{n+1}\cdot\frac{n}{(2x-5)^n}\right| \\ &= \lim_{n\to\infty}\left|(2x-5)\frac{n}{n+1}\right| \\ &= |2x-5|.\end{aligned}$$

By the ratio test, this converges when $|2x - 5| < 1$ (i.e., when $2 < x < 3$) and is inconclusive (may possibly converge) when $|2x - 5| = 1$ (i.e., when $x = 2, x = 3$). When $x = 2$, our series is the alternating harmonic series, and thus the series converges. When $x = 3$, our series is the harmonic series, which diverges.

The series converges for $2 \le x < 3$.

Example 8.5.3. For which values of x does the series converge? $\displaystyle\sum_{n=0}^{\infty} (n!)(x-1)^n$

Applying the ratio test, we have

$$\lim_{n\to\infty}\left|\frac{(n+1)!(x-1)^{n+1}}{n!(x-1)^n}\right| = \lim_{n\to\infty}|x-1||n+1|$$

When $x \neq 1$, this limit is infinite, and when $x = 1$, the limit is 0. Thus the series converges precisely when $x = 1$.

Theorem 8.5.4. *For given power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$*, there are only three possibilities:*

1. *The series converges only when* $x = a$*.*
2. *The series converges for all* x*.*
3. *There is a positive number* R *such that the series converges if* $|x-a| < R$ *and diverges if* $|x-a| > R$*.*

Definition. The number R in case 3 above is called the **radius of convergence** of the power series. The interval consisting of all values of x for which the power series converges is called the **interval of convergence**.

Remark. The interval of convergence is not just the open interval $(a-R, a+R)$, but may actually include the endpoints of this interval as well - these have to be tested separately. For example, Example 8.5.2 has interval of convergence $[2, 3)$.

Example 8.5.5. Find the radius and interval of the convergence for the following series: $\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$

Notice this is a geometric series with $a = 1$ and $r = \frac{x}{2}$. By the geometric series test, this converges precisely when $\left|\frac{x}{2}\right| < 1$, i.e., when $|x| < 2$. So the radius of convergence is 2 and the interval of convergence is $(-2, 2)$.

Example 8.5.6. Find the interval and radius of convergence for the following series: $\sum_{n=1}^{\infty} \frac{(x-4)^n}{\sqrt[3]{n}}$

Using the ratio test, we have

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{\frac{(x-4)^{n+1}}{\sqrt[3]{n+1}}}{\frac{(x-4)^n}{\sqrt[3]{n+1}}} \right| &= \lim_{n\to\infty} \left| \frac{(x-4)^{n+1}}{\sqrt[3]{n+1}} \cdot \frac{\sqrt[3]{n}}{(x-4)^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{(x-4)\sqrt[3]{n}}{\sqrt[3]{n+1}} \right| \\
&= \lim_{n\to\infty} |x-4| \cdot \left| \frac{\sqrt[3]{n}}{\sqrt[3]{n+1}} \right| \\
&= |x-4|
\end{aligned}$$

converges when this limit is less than 1, and so $|x-4| < 1$ tells us that the radius of convergence is 1. The *open interval of convergence* is thus $(4-1, 4+1) = (3, 5)$. Checking the endpoints, $x = 3$ converges by the alternating series test, and $x = 5$ diverges by the p-series test. So the interval of convergence is $[3, 5)$.

Example 8.5.7. Find the interval and radius of convergence for the following series: $\displaystyle\sum_{n=1}^{\infty} \frac{n^3(x+5)^n}{6^n}$

Using the ratio test, we have

$$\lim_{n\to\infty} \left| \frac{(n+1)^3(x+5)^{n+1}}{6^{n+1}} \cdot \frac{6^n}{n^3(x+5)^n} \right| = \left| \frac{x+5}{6} \right|$$

converges when this limit is less than 1, or equivalently, when $|x+5| < 6$, and so the radius of convergence is 6. To find the open interval of convergence, we have $|x+5| < 6$ implies $-11 < x < 1$. When $x = -11$ and 1, the series diverges by divergence test, so the interval of convergence $(-11, 1)$.

Example 8.5.8. Find the interval and radius of convergence for the following series: $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{\ln(n+4)}$

Using the ratio test, we have

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{\ln(n+5)} \cdot \frac{\ln(n+4)}{(x-2)^n} \right| = |x-2|$$

converges when this limit is less than 1, and so $|x-2| < 1$ tells us that the radius of convergence is 1. The open interval of convergence is thus $(2-1, 2+1) = (1, 3)$. Checking the endpoints, when $x = 1$ the series converges by alternating series test, and when $x = 3$ the series diverges by comparing to the series $\sum \frac{1}{n+4}$. Thus, the interval of convergence is $[1, 3)$.

8.6 Representing Functions as Power Series

We know that, when $|x| < 1$, we have

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

What this tells us is that the function $f(x) = \frac{1}{1-x}$ can be approximated to any degree of accuracy (for values of x with $-1 < x < 1$ anyway) by just looking at polynomials $1 + x + x^2 + \cdots + x^n$ for as large n as we require. This is fantastic as polynomials are well-studied and extremely easy to evaluate.

Example 8.6.1. Express $f(x) = \frac{1}{1+8x^3}$ as a power series and find its interval of convergence.

Notice that we have

$$f(x) = \frac{1}{1+8x^3} = \frac{1}{1-(-8x^3)} = \sum_{n=0}^{\infty}(-8x^3)^n = \sum_{n=0}^{\infty}(-1)^n(8x^3)^n.$$

This converges when $|8x^3| < 1$, i.e., when $|x| < \frac{1}{2}$. The interval of convergence is thus $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Theorem 8.6.2. *If the power series* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *has radius of convergence* $R > 0$*, then the function* $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ *is differentiable on the interval* $(a-R, a+R)$ *and*

1. $\dfrac{d}{dx}f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{d}{dx}\left[c_n(x-a)^n\right] = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$
2. $\displaystyle\int f(x)\,dx = \sum_{n=0}^{\infty}\int c_n(x-a)^n\,dx = C + \sum_{n=0}^{\infty}\frac{c_n}{n+1}(x-a)^{n+1}$

Example 8.6.3. Given the power series for $f(x) = \dfrac{1}{1-x}$, use differentiation to express $g(x) = \dfrac{1}{(1-x)^2}$ as a power series. Find the radius of convergence of this new power series.

We notice that

$$f'(x) = \frac{1}{(1-x)^2} = g(x),$$

and so

$$g(x) = \frac{d}{dx}f(x) = \sum_{n=0}^{\infty}\frac{d}{dx}\left[x^n\right] = \sum_{n=1}^{\infty} nx^{n-1}.$$

Using the Ratio test, we have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)x^n}{nx^{n-1}}\right| \\ &= |x|,\end{aligned}$$

which converges when $|x| < 1$, so the radius of convergence is 1, which is exactly the same as the radius of convergence for the series representation of $f(x)$.

Proposition 8.6.4. *Given the series with radius of convergence $R > 0$ and the function $f(x)$ defined in the premise for Theorem 8.6.2, the new series obtained from $\frac{d}{dx}[f(x)]$ and $\int f(x)\,dx$ both have radius of convergence R.*

Example 8.6.5. Given the power series for $f(x) = \dfrac{1}{1+x}$, find a power series representation for $g(x) = \ln(1+x)$.

Notice that

$$g'(x) = \frac{1}{1+x} = f(x),$$

so we have

$$\begin{aligned} g(x) = \int f(x)\,dx = \int \frac{1}{1-(-x)}\,dx &= \int \left[\sum_{n=0}^{\infty}(-1)^n x^n\right] dx \\ &= \sum_{n=0}^{\infty}\int (-1)^n x^n\,dx \\ &= C + \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1}. \end{aligned}$$

To determine C, we set $x = 0$ and get that $g(0) = 0 = C$, so our power series representation is just

$$g(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n.$$

Example 8.6.6. Find a power series representation for $f(x) = \arctan(x)$

Notice that

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)},$$

so we have that

$$\begin{aligned} f(x) = \int \frac{1}{1-(-x^2)}\,dx &= \int \left[\sum_{n=0}^{\infty}(-1)^n x^{2n}\right] dx \\ &= \sum_{n=0}^{\infty}\int (-1)^n x^{2n}\,dx \\ &= C + \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}. \end{aligned}$$

To find C, we set $x = 0$ and get that $f(0) = 0 = C$, so the power series representation is just

$$f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}.$$

Example 8.6.7. The function $f(x) = \dfrac{4}{(2-x)^2}$ is the derivative of $g(x) = \dfrac{2x}{2-x}$. Find a power series representation for $f(x)$.

$$g(x) = \frac{2x}{2-x} = \frac{2x}{2}\left(\frac{1}{1-\frac{1}{2}x}\right) = \frac{2x}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}x\right)^n = \sum_{n=0}^{\infty}\frac{x^{n+1}}{2^n}.$$

Differentiating this,

$$\begin{aligned} f(x) = \frac{d}{dx}[g(x)] &= \frac{d}{dx}\left[\sum_{n=0}^{\infty}\frac{x^{n+1}}{2^n}\right] \\ &= \sum_{n=0}^{\infty}\frac{d}{dx}\left[\frac{x^{n+1}}{2^n}\right] \\ &= \sum_{n=0}^{\infty}\frac{(n+1)x^n}{2^n} \end{aligned}$$

Example 8.6.8. Find a power series representation for $f(x) = e^x$.

We know that know that $f'(x) = f(x) = e^x$. So, we can write

$$e^x = f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

$$e^x = f'(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n x^n\right] = \sum_{n=1}^{\infty} c_n n x^{n-1} = c_1 + c_2 x + c_3 x^2 + \cdots$$

And so,

$$\begin{aligned} c_0 &= c_1 \\ c_1 &= 2c_2 \\ c_2 &= 3c_3 \\ &\vdots \\ c_{n-1} &= nc_n \end{aligned}$$

We have that $1 = f(0) = c_0$, and so using this with the above equalities to compute $c_1 = 1$, $c_2 = \frac{1}{2}$, etc. we see $c_n = \frac{1}{n!}$. So series representation is

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

8.7 Taylor and Maclaurin Series

Suppose that for $|x-a|<R$, $f(x)$ has the power series representation

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

We notice that $f(a) = c_0$, so maybe we can write all of the other coefficients in terms of f. Indeed,

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-)^3 + \cdots$$

Now we have that $f'(a) = c_1$. Again,

$$f''(x) = 2c_2 + 6c_3(x-a) + 24c_4(x-a)^2 + 120c_5(x-a)^3 + \cdots$$

Now we have that $\dfrac{f''(a)}{2} = c_2$. Iterating through consecutive derivatives (and adopting the convention that $f^{(0)}(x) \equiv f(x)$, we get the following relationship for the coefficients:

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Definition. The **Taylor series for the function f centered at a** is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The **Maclaurin series for the function f** is the Taylor series with $a=0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

Remark. Our derivation relies on the fact that $f(x)$ has a power series representation. If it does not, it does not need to be the sum of the Taylor series.

Example 8.7.1. Find the Maclaurin series of the function $f(x) = e^x$ and the radius of convergence.

Since $f^{(n)}(x) = e^x$ for all n, $f^{(n)}(0) = 1$ for all n. Thus we get

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

This matches what we saw in Example 8.6.8 By the ratio test, we can see that it has infinite radius of convergence (the series converges for all x-values).

Example 8.7.2. Find the Taylor series expansion of the function $f(x) = \ln(x)$ centered at $a = 2$.

$$\begin{aligned}
f^{(0)}(a) &= \ln a \\
f^{(1)}(a) &= \frac{1}{a} \\
f^{(2)}(a) &= -\frac{1}{a^2} \\
f^{(3)}(a) &= \frac{2}{a^3} \\
f^{(4)}(a) &= -\frac{6}{a^4} \\
&\vdots \\
f^{(n)}(a) &= \frac{(-1)^{n-1}(n-1)!}{a^n} && \text{(for } n > 0\text{)}
\end{aligned}$$

Seeing this pattern, we notice that $c_0 = \ln 2$, but every other coefficient has the form

$$c_n = \frac{f^{(n)}(2)}{n!} = \frac{(-1)^{n-1}(n-1)!}{n!2^n} = \frac{(-1)^{n-1}}{n2^n}.$$

Thus the Taylor series expansion is

$$\ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n}(x-2)^n$$

Example 8.7.3. Find the Maclaurin series expansion of the function $f(x) = \sin x$.

$$\begin{aligned}
f^{(0)}(0) &= \sin 0 = 0 \\
f^{(1)}(0) &= \cos 0 = 1 \\
f^{(2)}(0) &= -\sin 0 = 0 \\
f^{(3)}(0) &= -\cos 0 = -1 \\
f^{(4)}(0) &= \sin 0 = 0 \\
f^{(5)}(0) &= \cos 0 = 1 \\
&\vdots
\end{aligned}$$

So, we notice that the Maclaurin series for $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Example 8.7.4. Find the Maclaurin series expansion of the function $g(x) = \cos x$.

Since $g(x) = f'(x)$, we can differentiate the Maclaurin series for $\sin x$ from the previous example to get

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Example 8.7.5. Let i be the imaginary unit (so $i^2 = -1$). What sort of relationship do you notice about the Maclaurin series expansions for e^{ix}, $\cos x$, and $\sin x$?

This exploration is left to the reader.

Example 8.7.6. Find the Maclaurin series expansion of the binomial $f(x) = (1+x)^k$ for some fixed number k.

$$\begin{aligned}
f^{(0)}(0) &= (1+x)^k = 1\\
f^{(1)}(0) &= k(1+x)^{k-1} = k\\
f^{(2)}(0) &= k(k-1)(1+x)^{k-2} = k(k-1)\\
f^{(3)}(0) &= k(k-1)(k-2)(1+x)^{k-3} = k(k-1)(k-2)\\
&\vdots\\
f^{(n)}(0) &= k(k-1)(k-2)\cdots(k-n+1).
\end{aligned}$$

So the Maclaurin series for $(1+x)^k$ is

$$\begin{aligned}
(1+x)^k &= 1 + kx + \frac{k(k-1)}{2}x^2 + \frac{k(k-1)(k-2)}{6}x^3 + \cdots\\
&= \sum_{n=0}^{\infty} \binom{k}{n} x^n.
\end{aligned}$$

where here $\binom{k}{n}$ is the notation given to the coefficients. We note that when k and n are both positive integers satisfying $k \geq n$, then $\binom{k}{n}$ is exactly the same as what you may have seen in the contexts of combinatorics (and thus $\binom{k}{n} = 0$ for $n > k$).

Example 8.7.7. Find the Maclaurin series for $\dfrac{2\sin(3x)}{x}$.

We can make simple modifications to the Maclaurin series for $\sin x$.

$$\begin{aligned}
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\\
\sin(3x) &= \sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n+1}}{(2n+1)!}\\
2\sin(3x) &= 2\sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n+1}}{(2n+1)!}\\
\frac{2\sin(3x)}{x} &= \frac{1}{x} \cdot 2\sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n+1}}{(2n+1)!} = 2\sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1}x^{2n}}{(2n+1)!}
\end{aligned}$$

To recap, below is a list of particularly useful and common Maclaurin series and radii of convergence.

Function	**Maclaurin Series**	**Radius of Convergence**
$\dfrac{1}{1-x}$	$\displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$	$R = 1$
$\ln(1+x)$	$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$R = 1$
e^x	$\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$	$R = \infty$
$\sin x$	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$R = \infty$
$\cos x$	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$R = \infty$
$\arctan x$	$\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$R = 1$
$(1+x)^k$	$\displaystyle\sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \cdots$	$R = \infty$ if k is a nonnegative integer, $R = 1$ otherwise.

8.7.1 Taylor Polynomials

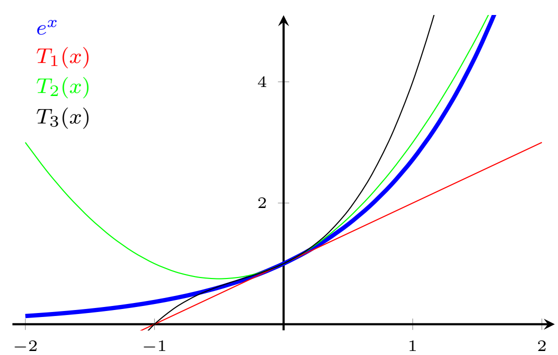
Definition. The **k^{th} Taylor polynomial for $f(x)$ centered at a** is the k^{th} partial sum of the Taylor series:

$$T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Notice that when $k = 1$, we have the function

$$T_k(x) = f(a) + f'(a)(x-a).$$

This is just the tangent line approximation of the function f at a! It suggests to us that, as k grows, the k^{th} Taylor polynomial provides us a better and better approximation of our function values when x is close to a. Indeed, if we look at the graph of $y = e^x$ below and a few Taylor polynomials (centered at 0).



Just as with a tangent line approximation, the center a is important, as your estimation becomes less accurate if x and a are very far away.

If $f(x)$ is the sum of its Taylor series and $T_k(x)$ is an approximation, then the difference $R_k(x) = f(x) - T_k(x)$ should tell us the error involved in estimating the value of the function. Indeed, we call $R_k(x)$ the **remainder** of the Taylor series, and $|R_k(x)|$ is the **(absolute value of the) error**.

Example 8.7.8. Use a fifth-degree Taylor polynomial to approximate $\sin(3.15)$. Find the error of the approximation $\sin(3.15) \approx T_5(3.15)$.

Since 3.15 is very close to π, we'll center the Taylor polynomial at $a = \pi$.

$$\begin{aligned}
f^{(0)}(\pi) &= 0 \\
f^{(1)}(\pi) &= -1 \\
f^{(2)}(\pi) &= 0 \\
f^{(3)}(\pi) &= 1 \\
f^{(4)}(\pi) &= 0 \\
f^{(5)}(\pi) &= -1 \\
\Rightarrow T_5(x) &= -(x-\pi) + \frac{(x-\pi)^3}{3!} - \frac{(x-\pi)^5}{5!}
\end{aligned}$$

So then

$$\sin(3.15) \approx T_5(3.15) = -0.008\,407\,247\,367\,148\,707\ldots$$

Checking with our calculator, we have

$$\sin(3.15) = -0.008\,407\,247\,367\,148\,706\ldots$$

and so the error is $|R_5(3.15)| \approx 1.22258 \times 10^{-16}$.

Another extremely important reason for using these power series (and Taylor series in particular) is because functions may not have antiderivatives that can be expressed in terms of elementary functions (simple addition, subtraction, exponents, etc). The following example is a very well-known example of such a function.

Example 8.7.9. Use the Maclaurin series to evaluate $\int e^{-t^2}\,dt$.

Recalling the Maclaurin series for e^x, we have

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}.$$

Thus

$$\int e^{-t^2}\,dt = \int \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}\,dt = \sum_{n=0}^{\infty} \int \frac{(-1)^n t^{2n}}{n!}\,dt = C + \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)n!}.$$

Example 8.7.10. We define the (unnormalized) *error function* $\operatorname{erf}(x)$ to be

$$\operatorname{erf}(x) = \int_0^x e^{-t^2}\,dt$$

Use a fifth-degree Maclaurin polynomial to approximate $\operatorname{erf}(1)$.

From the previous example, we have that

$$\begin{aligned} erf(x) &= \left[C + \sum_{n=1}^{\infty} \frac{(-1)^n t^{2n+1}}{n!(2n+1)} \right]_0^x \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} \end{aligned}$$

Thus, the fifth-order Maclaurin polynomial is

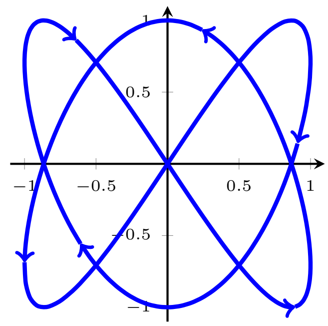
$$T_5(x) = x - \frac{x^3}{3} + \frac{x^5}{10}$$

and our approximation of $\operatorname{erf}(1)$ is

$$\operatorname{erf}(1) \approx T_5(1) = 1 - \frac{1}{3} + \frac{1}{10} \approx 0.767$$

9 Parametric Equations and Polar Coordinates

9.1 Parametric Curves

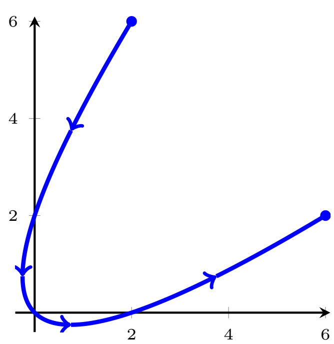


A parametric curve C given by $x = \sin(2t)$, $y = \sin(3t)$.

The graph of the curve above fails to be a function of the form $y = f(x)$, because it fails the vertical line test, but it may be a reasonable path for an object to travel (maybe a weight attached to a spring attached to a pendulum, or maybe a bee's flight path), so we'd like to be able to model it.

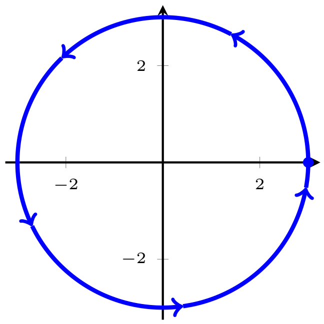
Suppose x and y are both functions of a third variable, t (called a **parameter**), with $x = f(t)$ and $y = g(t)$ (called **parametric equations**). We can then plot the points $(x, y) = (f(t), g(t))$ in the coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ traces out a curve C (called a **parametric curve**).

Example 9.1.1. Sketch the curve given by $x = t^2 + t$, $y = t^2 - t$, $-2 \leq t \leq 2$. Indicate with an arrow the direction in which the curve is traced as t increases.



t	(x, y)
-2	$(2, 6)$
-1	$(0, 2)$
0	$(0, 0)$
1	$(2, 0)$
2	$(6, 2)$

Example 9.1.2. Sketch the curve given by $x = 3\cos t$, $y = 3\sin t$, for $0 \le t \le 2\pi$. Indicate with an arrow the direction in which the curve is traced as t increases.



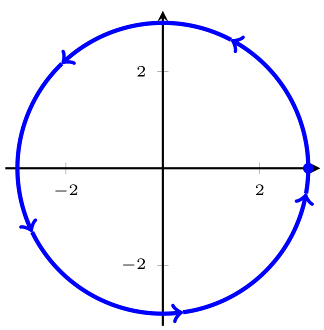
t	(x, y)
0	$(3, 0)$
$\frac{\pi}{2}$	$(0, 3)$
π	$(-3, 0)$
$\frac{3\pi}{2}$	$(0, -3)$
2π	$(3, 0)$

The shape appears to be a circle of radius 3. And indeed, we can see this is the case be eliminating the parameter:

$$x^2 + y^2 = 9\cos^2 t + 9\sin^2 t = 9(\sin^2 t + \cos^2 t) = 9,$$

so the equation is exactly that of a circle of radius 3.

Example 9.1.3. Sketch the curve given by $x = 3\cos(2t)$, $y = 3\sin(2t)$, for $0 \le t \le \pi$. Indicate with an arrow the direction in which the curve is traced as t increases.



t	(x, y)
0	$(3, 0)$
$\frac{\pi}{2}$	$(0, 3)$
π	$(-3, 0)$
$\frac{3\pi}{2}$	$(0, -3)$
2π	$(3, 0)$

Once again, the shape appears to be a circle of radius 3. And indeed, we can see this is the case be eliminating the parameter:

$$x^2 + y^2 = 9\cos^2(2t) + 9\sin^2(2t) = 9,$$

so the equation is exactly that of a circle of radius 3.

What this example illustrates is that *curves are not uniquely parametrized*. If thinking about a particle traveling along these curves, the particle in the second example completes the curve in half the amount of time (or rather, travels twice as fast).

Because curves are not uniquely parametrized, it may be easier to visualize the curve by eliminating the parameter and obtaining a Cartesian equation of the curve.

Example 9.1.4. Eliminate the parameter from

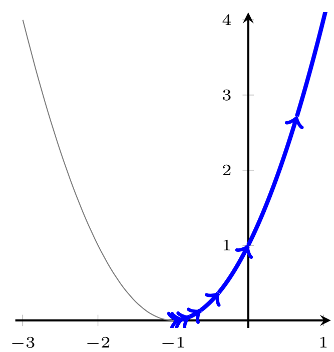
$$x = e^t - 1,\ y = e^{2t}$$

to find a Cartesian equation of the curve. Then sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

By rearranging the first equation as $e^t = x + 1$, we have

$$y = e^{2t} = (e^t)^2 = (x+1)^2,$$

which is a parabola. One thing to keep in mind is that $e^t > 0$ for all t, so the range of x-values is the interval $(1, \infty)$. Since x is increasing as t increases, the arrows trace the curve from left to right.



Example 9.1.5. Eliminate the parameter from

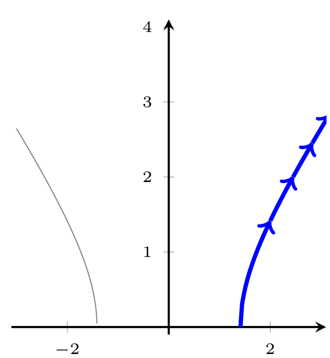
$$x = \sqrt{t+1},\ y = \sqrt{t-1}$$

to find a Cartesian equation of the curve. Then sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

By rearranging the first equation as $t = x^2 - 1$ and the second as $t = y^2 + 1$ we have

$$x^2 - 1 = y^2 + 1 \quad \Rightarrow \frac{x^2}{2} - \frac{y^2}{2} = 1,$$

which is a hyperbola. One thing to keep in mind is that the arguments for $x(t)$ and $y(t)$ are only defined for $t \geq 1$, so the range of x-values is the interval $(2, \infty)$ and the range of y-values is $(0, \infty)$. Since x is increasing as t increases, the arrows trace the curve from left to right.



9.2 Calculus with Parametric Curves

9.2.1 Slopes and Tangent Lines

Suppose that f and g are differentiable functions and we have the parametric curve C given by $x(t) = f(t)$ and $y(t) = g(t)$ (where y can also be expressed as a differentiable function of x). If we want to find the tangent line at a point (x, y) on C, we need to find $\frac{dy}{dx}$. By an application of the chain rule, we have

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

which rearranges to

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \quad \text{if } \frac{dx}{dt} \neq 0. \tag{9.2.1}$$

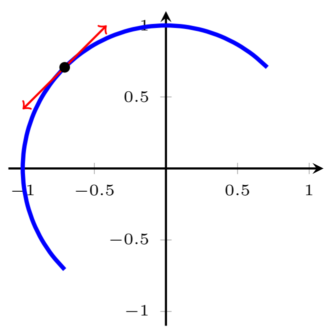
Remark. While $\frac{dy}{dx}$ has a simple expression in terms of parametric derivatives, it is not quite so straightforward to find higher derivatives. To find $\frac{d^2y}{dx^2}$ we rewrite Equation 9.2.1 as

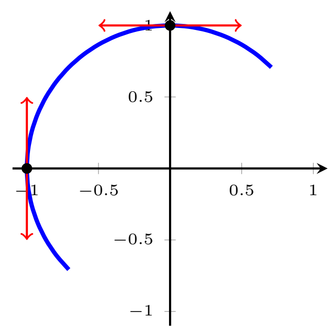
$$\frac{d}{dx}(y) = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(y)}{\frac{dx}{dt}}$$

and then replacing y with $\frac{dy}{dx}$ yields

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)}. \tag{9.2.2}$$

Example 9.2.1. Find the equation of the tangent line through the point where $t = \frac{3}{4}$ of the parametric curve given by $x = \cos(\pi t)$, $y = \sin(\pi t)$, $\frac{1}{4} \leq t \leq \frac{5}{4}$. At what point is the slope horizontal? At what point is the slope vertical?





We have that $\frac{dx}{dt} = -\pi \sin(\pi t)$ and $\frac{dy}{dt} = \pi \cos(\pi t)$, so the slope of the tangent line is

$$\left.\frac{dy}{dx}\right|_{t=\frac{3}{4}} = \frac{\pi \cos\left(\frac{3\pi}{4}\right)}{-\sin\left(\frac{3\pi}{4}\right)} = 1$$

and passes through the point $\left(x(\frac{3}{4}), y(\frac{3}{4})\right) = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, so the equation for this line is

$$y = x + \sqrt{2}.$$

Recall that the tangent line is horizontal if the slope is 0, i.e., if $\left.\frac{dy}{dt}\right|_t = 0$, and vertical if $\left.\frac{dx}{dt}\right|_t = 0$. Thus we have a horizontal tangent line when $t = \frac{1}{2}$ and a vertical tangent line when $t = 1$.

Example 9.2.2. For the parametric curve given by $x = \sqrt{t}$, $y = \frac{1}{4}(t^2 - 4)$, $t \geq 0$, find the slope and concavity at the point $(2, 3)$.

We have that $\frac{dx}{dt} = \frac{1}{2\sqrt{t}}$ and $\frac{dy}{dt} = \frac{t}{2}$. Note also that $(x(t), y(t)) = (2, 3)$ when $t = 4$. So, the first derivative is

$$\frac{dy}{dx} = \frac{\frac{t}{2}}{\frac{1}{2\sqrt{t}}} = t^{3/2}.$$

and the slope at the point where $t = 4$ is

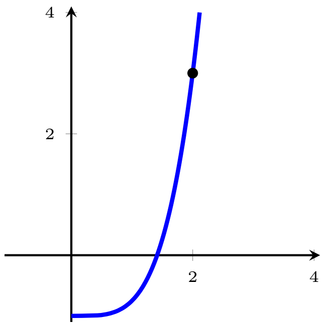
$$\left.\frac{dy}{dx}\right|_{t=4} = 4^{3/2} = 8.$$

The second derivative is

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[t^{3/2}\right]}{\frac{1}{2\sqrt{t}}} = \frac{\frac{3\sqrt{t}}{2}}{\frac{1}{2\sqrt{t}}} = 3t,$$

and the concavity at this point where $t = 4$ is thus

$$\left.\frac{d^2y}{dx^2}\right|_{t=4} = 3(4) = 12$$



Example 9.2.3. The *prolate cycloid* given by $x = 2t - \pi \sin t$ and $y = 2 - \pi \cos t$ crosses itself at the point $(0, 2)$. Find the equations of both tangent lines at this point.

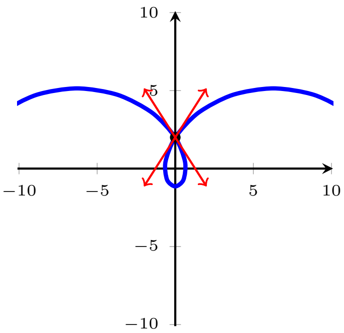
We find that $(x(t), y(t)) = (0, 2)$ when $t = -\frac{\pi}{2}$ and $t = \frac{\pi}{2}$. We have that $\frac{dx}{dt} = 2 - \pi \cos t$ and $\frac{dy}{dx} = \pi \sin t$. The slope at $t = -\frac{\pi}{2}$ is thus

$$\left.\frac{dy}{dx}\right|_{t=-\pi/2} = \frac{\pi \sin\left(-\frac{\pi}{2}\right)}{2 - \pi \cos\left(-\frac{\pi}{2}\right)} = -\frac{\pi}{2}$$

and the equation of the tangent line here is $y = -\frac{\pi}{2}x + 2$. The slope at $t = \frac{\pi}{2}$ is

$$\left.\frac{dy}{dx}\right|_{t=-\pi/2} = \frac{\pi \sin\left(\frac{\pi}{2}\right)}{2 - \pi \cos\left(\frac{\pi}{2}\right)} = \frac{\pi}{2}$$

and the equation of the tangent line here is $y = \frac{\pi}{2}x + 2$.



9.2.2 Arc Length

We saw how to compute arc length in Section 7.4. We can equivalently discuss arc length in terms of parametric functions thanks to the following theorem

Theorem 9.2.4. *If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$ where f' and g' are continuous on $[\alpha, \beta]$ and C is transversed exactly once on this interval, then the length of C is*

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{\alpha}^{\beta} \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

Proof. See the text. $\square$

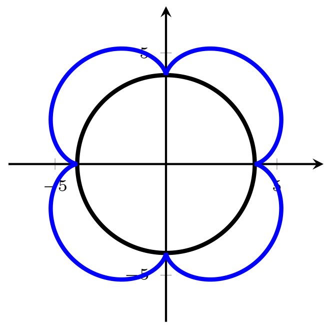
Example 9.2.5. Let $r > 0$ be some real number and consider the curve given by $x = r\cos t$, $y = r\sin t$. Find the arc length of this curve for $0 \leq t \leq 2\pi$.

Applying the arc length formula,

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt \\ &= \int_0^{2\pi} r\, dt \\ &= [rt]_0^{2\pi} = 2\pi r. \end{aligned}$$

This is just the formula for the circumference of a circle of radius r, as expected.

Example 9.2.6. A circle of radius 1 rolls around the circumference of a larger circle of radius 4. The epicycloid traced by a point on the circumference of the smaller circle is given by $x = 5\cos t - \cos(5t)$, $y = 5\sin t - \sin(5t)$. Find the distance traveled by the point in one complete trip about the larger circle.



We appeal to symmetry and integrate the first quadrant's curve only, and then multiply the answer by 4:

$$\begin{aligned}
L &= 4\int_0^{\pi/2} \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt \\
&= 4\int_0^{\pi/2} \sqrt{[-5\sin t + 5\sin(5t)]^2 + [5\cos t - 5\cos(5t)]^2} \\
&= 4\int_0^{\pi/2} \sqrt{25 - 50\sin t\sin(5t) + 25 - 50\cos t\cos(5t)}\,dt \\
&= 20\int_0^{\pi/2} \sqrt{2 - 2\sin t\sin(5t) - 2\cos t\cos(5t)}\,dt \\
&= 20\int_0^{\pi/2} \sqrt{2 - 2\cos(4t)}\,dt && \text{(angle sum identity)} \\
&= 20\int_0^{\pi/2} \sqrt{4\sin^2(2t)}\,dt && \text{(double-angle identity)} \\
&= 20\int_0^{\pi/2} 2\sin(2t)\,dt \\
&= -20\left[\cos(2t)\right]_0^{\pi/2} = 40.
\end{aligned}$$

9.2.3 Areas

Recall that the area under a curve $y = F(x)$ from $x = a$ to $x = b$ is given by

$$A = \int_a^b y\,dx = \int_a^b F(x)\,dx.$$

If our curve is parametrized as $x = f(t)$, $y = g(t)$, and we have that $\alpha \le t \le \beta$ with $a = f(\alpha)$ and $b = f(\beta)$, then

$$dx = f'(t)\,dt$$

and the substitution rule for definite integrals gives us the area under a parametric curve as

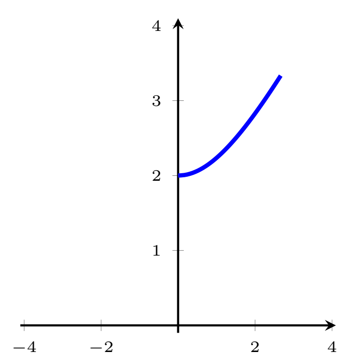
$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)f'(t)\,dt.$$

If instead we have that $a = f(\beta)$ and $b = f(\alpha)$ then the area under this parametric curve is

$$\int_\beta^\alpha g(t)f'(t)\,dt.$$

It's worth noting that we have to make some additional assumptions for this area to be unambiguous and for the integral to work. In particular, we need to make sure that the curve is traced out only once on this interval (otherwise we could be adding/subtracting extra area that isn't really there), and we also need to make sure that the curve doesn't fail the vertical line test on this interval.

Example 9.2.7. Find the area below the parametric curve $x = t - \frac{1}{t}$, $y = t + \frac{1}{t}$, $1 \le t \le 3$.

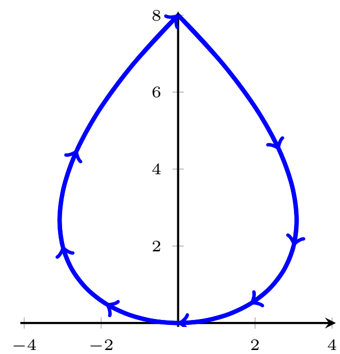


Since $x'(t), y'(t)$ are not 0 on this interval, the curve never traces itself out more than once. Plotting the portion of curve above, we can see that it also passes the vertical line test. So we integrate according to the definition above. Since $x'(t) = 1 + \frac{1}{t^2}$, we get

$$\begin{aligned} A &= \int_1^3 \left(t + \frac{1}{t}\right)\left(1 + \frac{1}{t^2}\right) dt \\ &= \int_1^3 t + \frac{2}{t} + \frac{1}{t^3}\,dt \\ &= \left[\frac{1}{2}t^2 + 2\ln|t| - \frac{1}{2t^2}\right]_1^3 \\ &= \frac{40}{9} + \ln 9 \approx 6.6417. \end{aligned}$$

If we relax the condition that our curve C pass the vertical line test and assume that C is a closed loop, then we can actually compute the area enclosed by the loop.

Example 9.2.8. Compute the area of the loop enclosed by the curve $x = t^3 - 4t$, $y = 2t^2$ on the interval $-2 \le t \le 2$.



Notice that the "upper half" of this curve is traversed from left to right as t increases. This means that integrating from $t = -2$ to $t = 2$ will result in a positive area. So, computing $x'(t) = 2t^2 - 4$, we get

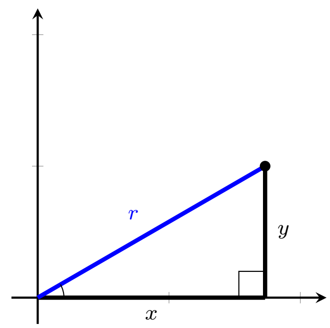
$$\begin{aligned} A &= \int_{-2}^{2} (2t^2)(2t^2 - 4)\, dt \\ &= \int_{-2}^{2} 4t^4 - 8t^2\, dt \\ &= \left[\frac{4}{5}t^5 - \frac{8}{3}t^3\right]_{-2}^{2} \\ &= \frac{128}{15} \approx 8.5333 \end{aligned}$$

Example 9.2.9. Compute the area of the circle of radius 1, centered at $(x, y) = (2, 2)$, using the parameterization $x = 2 + \cos(t)$, $y = 2 + \sin(t)$.

This exercise is left to the reader. We note, however, that the "upper half" of the circle is traversed backwards (that is, x is decreasing), and so you should choose your limits of integration accordingly.

9.3 Polar Coordinates

Let (x, y) be some point in the Cartesian plane and let r be the length of the line segment from the origin $(0, 0)$ to (x, y). Also, let θ be the angle from the positive x-axis to this line segment (traversing counter-clockwise). It's not hard to see that, when (x, y) is not the origin, this r-value and this θ-value are unique to this (x, y) point, so rather than refer to the point in terms of the ordered pair (x, y), we could refer to them in terms of (r, θ). This is the basis for **polar coordinates**, which is a very useful re-parametrization of the Cartesian plane. So how do we pass between them?



From trigonometry, we see that we have the following relationships

$$x = r\cos\theta, \qquad r = \sqrt{x^2 + y^2},$$
$$y = r\sin\theta, \qquad \tan\theta = \frac{y}{x}.$$

Remark. Since $\arctan x$ only has range $(-\frac{\pi}{2}, \frac{\pi}{2})$, you have to be aware of the quadrant in which your point (x, y) lies and may have to add multiples of π to get the correct angle.

Example 9.3.1. Convert the point $(1, \frac{\pi}{2})$ from polar coordinates to Cartesian coordinates.

We have that

$$x = r\cos\theta = 1\cos\left(\frac{\pi}{2}\right) = 0,$$
$$y = r\sin\theta = 1\sin\left(\frac{\pi}{2}\right) = 1,$$

so, in Cartesian coordinates, we have $(0, 1)$.

Example 9.3.2. Convert the point $(-\sqrt{3}, -1)$ from Cartesian coordinates to polar coordinates.

We have that

$$r = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2,$$
$$\tan\theta = \frac{-1}{-\sqrt{3}} \Rightarrow \theta = \frac{7\pi}{6},$$

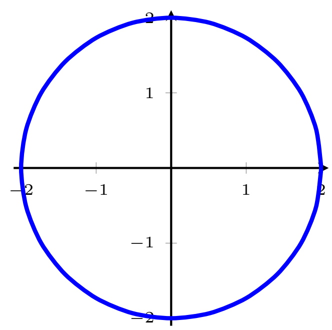
so, in polar coordinates, we have $\left(2, \frac{7\pi}{6}\right)$.

Definition. A **polar function** is a function of the form $r = f(\theta)$.

We graph the polar function $r = f(\theta)$ is the same we might graph $y = f(x)$: plot all values (r, θ) for which $r = f(\theta)$.

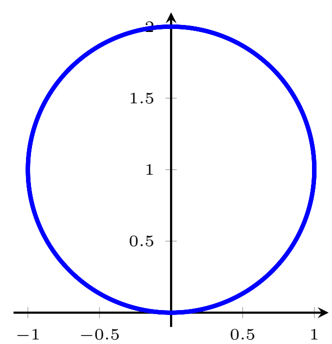
Example 9.3.3. Sketch a graph of the function $r = 2$.

θ	$x = r\cos(\theta)$	$y = r\sin(\theta)$
0	2	0
$\frac{\pi}{4}$	$\sqrt{2}$	$\sqrt{2}$
$\frac{\pi}{2}$	0	2
$\frac{3\pi}{4}$	$-\sqrt{2}$	$\sqrt{2}$
π	-2	0
$\frac{5\pi}{4}$	$-\sqrt{2}$	$-\sqrt{2}$
$\frac{3\pi}{2}$	0	-2
$\frac{7\pi}{4}$	$\sqrt{2}$	$-\sqrt{2}$



Example 9.3.4. Sketch a graph of the polar function $r = 2\sin\theta$. Find a Cartesian equation for this curve.

θ	$x = r\cos(\theta)$	$y = r\sin(\theta)$
0	0	0
$\frac{\pi}{4}$	1	1
$\frac{\pi}{2}$	0	2
$\frac{3\pi}{4}$	-1	1
π	0	0
$\frac{5\pi}{4}$	1	1
$\frac{3\pi}{2}$	0	2
$\frac{7\pi}{4}$	-1	1



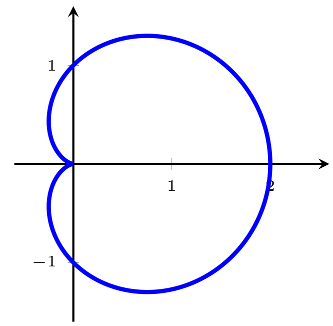
By some clever rearranging,

$$\begin{aligned} r &= 2\sin\theta \\ r^2 &= 2r\sin\theta \\ r^2 &= 2y \\ x^2 + y^2 &= 2y \\ x^2 + y^2 - 2y + 1 &= 1 \\ x^2 + (y-1)^2 &= 1 \end{aligned}$$

we get the equation for the circle of radius 1, centered at $(0, 1)$.

Example 9.3.5. Sketch a graph of the polar function $r = 1 + \cos\theta$.

θ	$x = r\cos(\theta)$	$y = r\sin(\theta)$
0	2	0
$\frac{\pi}{4}$	$\frac{1}{2} + \frac{\sqrt{2}}{2}$	$\frac{1}{2} + \frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$	0	1
$\frac{3\pi}{4}$	$\frac{1}{2} - \frac{\sqrt{2}}{2}$	$-\frac{1}{2} + \frac{\sqrt{2}}{2}$
π	0	0
$\frac{5\pi}{4}$	$\frac{1}{2} - \frac{\sqrt{2}}{2}$	$\frac{1}{2} - \frac{\sqrt{2}}{2}$
$\frac{3\pi}{2}$	0	-1
$\frac{7\pi}{4}$	$\frac{1}{2} + \frac{\sqrt{2}}{2}$	$-\frac{1}{2} - \frac{\sqrt{2}}{2}$

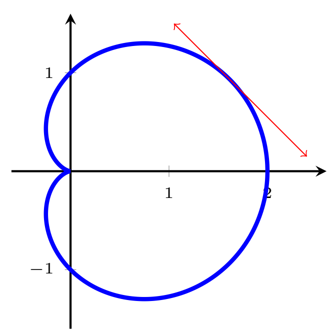


9.3.1 Tangents to Polar Curves

If $r = f(\theta)$, then we can regard θ as a parameter and we get

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}[r\sin\theta]}{\frac{d}{d\theta}[r\cos\theta]} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

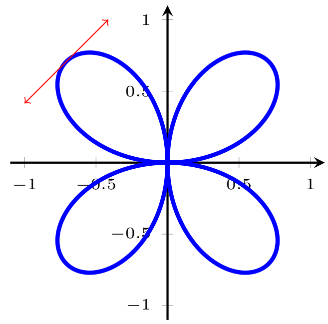
Example 9.3.6. Find the slope of the tangent line to $r = 1 + \cos\theta$ when $\theta = \frac{\pi}{6}$.



We have that $\frac{dr}{d\theta} = -\sin\theta$, so the slope of the tangent line when $\theta = \frac{\pi}{6}$ is

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \frac{-\sin(\frac{\pi}{6})\sin(\frac{\pi}{6}) + (1+\cos(\frac{\pi}{6}))\cos(\frac{\pi}{6})}{-\sin(\frac{\pi}{6})\cos(\frac{\pi}{6}) - (1+\cos(\frac{\pi}{6}))\sin(\frac{\pi}{6})} = -1.$$

Example 9.3.7. Find the equation of the tangent line to $r = \sin(2\theta)$ at $\theta = \frac{-\pi}{4}$.



We have that $\frac{dr}{d\theta} = 2\cos(2\theta)$, so the slope of the tangent line when $\theta = -\frac{\pi}{4}$ is

$$\left.\frac{dy}{dx}\right|_{\theta=-\pi/4} = \frac{2\cos(-\frac{\pi}{2})\sin(-\frac{\pi}{4}) + \sin(-\frac{\pi}{2})\cos(-\frac{\pi}{4})}{2\cos(-\frac{\pi}{2})\cos(-\frac{\pi}{4}) - \sin(-\frac{\pi}{2})\sin(-\frac{\pi}{4})} = 1.$$

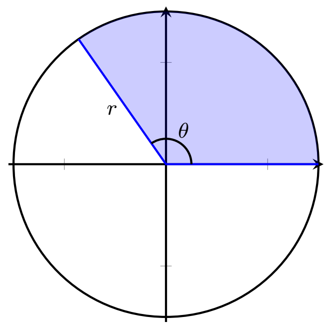
When $\theta = -\frac{\pi}{4}$, we have that $(x, y) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. Thus, the line with slope 1 passing through the point $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ has equation

$$y = x + \sqrt{2}$$

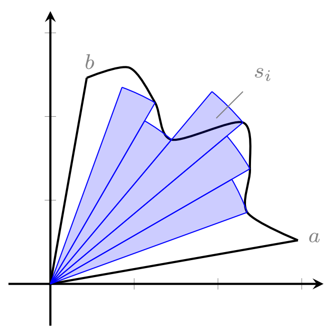
9.4 Areas and Lengths in Polar Coordinates

9.4.1 Area in Polar Coordinates

A sector of a circle of radius r spanned by angle θ has area $A = \pi r^2 \cdot \frac{\theta}{2\pi} = \frac{1}{2}r^2\theta$.



Just as we could use rectangles to approximate the area under the curve of a function in cartesian coordinates, we can use sectors to approximate the region enclosed by a polar curve.



Given the polar curve $r = f(\theta)$ from $\theta = a$ to $\theta = b$, we approximate the region with n sectors. The area of each sector s_i spanned by angle θ_i is $\frac{1}{2}[f(\theta_i)]^2\theta_i$. And thus the approximate area of the polar region is

$$A \approx \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i)]^2\theta_i$$

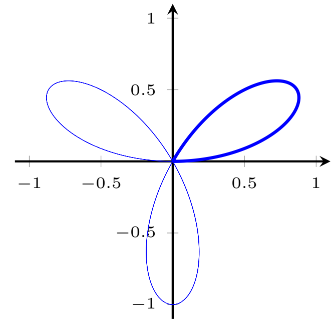
As n increases, the size of each sector decreases and our approximation gets better and better. Thus, we take a limit as $n \to \infty$ to get that our exact area is

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(\theta_i)]^2\theta_i = \int_a^b \frac{1}{2}[f(\theta)]^2\, d\theta.$$

Example 9.4.1. Find the area of one leaf of the rose $r = \sin(3\theta)$.

Notice that $0 \le \theta \le \frac{\pi}{3}$ traces out one leaf of the rose. We thus compute the area

$$\begin{aligned} A &= \int_0^{\pi/3} \frac{1}{2}\left[\sin(3\theta)\right]^2\, d\theta \\ &= \frac{1}{4}\int_0^{2} 1 - \cos(6\theta)\, d\theta \\ &= \frac{1}{4}\left[1 - \frac{1}{6}\sin(6\theta)\right]_0^{\pi/3} \\ &= \frac{\pi}{12}. \end{aligned}$$



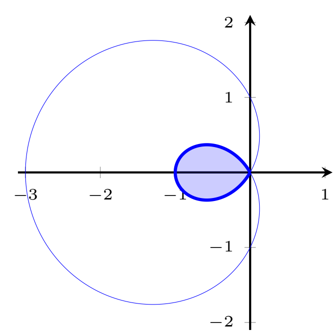
We always want to "sweep" the area counter-clockwise, so it's important that we choose our limits a and b so that $a \le \theta \le b$.

Example 9.4.2. Find the area of the inner loop of the limaçon $r = 1 - 2\cos\theta$.

The curve passes through itself at the origin, which corresponds to $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$. However, when we integrate, we need to choose the interval $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ to make sure we compute the correct region (compare Figure 9.4.2 and Figure 9.4.1).

As such, we have that the area is given by

$$\begin{aligned} A &= \int_{-\pi/3}^{\pi/3} \frac{1}{2}\left(1 - 2\cos\theta\right)^2\, d\theta \\ &= \int_{-\pi/3}^{\pi/3} \frac{1}{2} - 2\cos\theta + 2\cos^2\theta\, d\theta \\ &= \int_{-\pi/3}^{\pi/3} \frac{1}{2} - 2\cos\theta + \left(1 + \cos(2\theta)\right)\, d\theta \\ &= \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{2}\sin(2\theta)\right]_{-\pi/3}^{\pi/3} \\ &= \pi - \frac{3\sqrt{3}}{2}. \end{aligned}$$



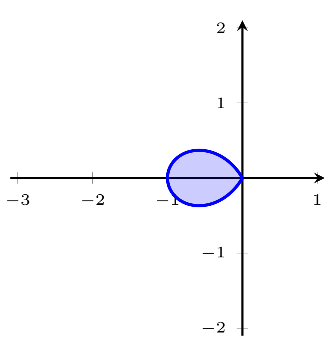


Figure 9.4.1: Area when integrating from $-\frac{\pi}{3}$ to $\frac{\pi}{3}$.

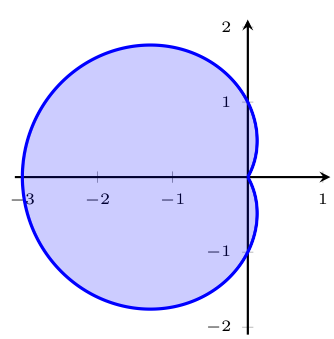
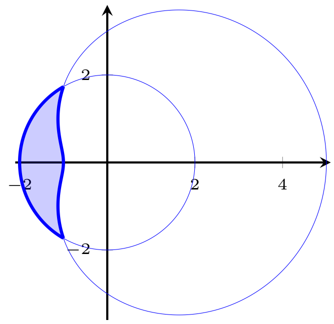


Figure 9.4.2: Area when integrating from $\frac{\pi}{3}$ to $\frac{5\pi}{3}$.

Just as in the case of Cartesian coordinates, the area in between two polar curves is the area of the outer polar region minus the area of the inner polar region. We note that we again have to verify that the angle measures in our limits of integration are correct for each curve separately.

Example 9.4.3. Find the area of the region inside the curve $r = 2$ and outside the curve $r = 3+2\cos\theta$.



The two curves intersect at $(r,\theta) = (2, \frac{2\pi}{3})$ and $(r,\theta) = (2, \frac{4\pi}{3})$. The area inside $r = 2$ from this region is given by

$$A_1 = \int_{2\pi/3}^{4\pi/3} \frac{1}{2}(2)^2\,d\theta = \frac{4\pi}{3}$$

and the area inside $r = 3 + 2\cos\theta$ from this region is given by

$$A_2 = \int_{2\pi/3}^{4\pi/3} \frac{1}{2}\left(3 + 2\cos\theta\right)^2\,d\theta = -\frac{11\sqrt{2}}{2} + \frac{11\pi}{3}.$$

Therefore the area of the region between the two curves is

$$A = A_1 - A_2 = -\frac{7\pi}{3} + \frac{11\sqrt{2}}{2}.$$

9.4.2 Arc Length in Polar Coordinates

Recall that for a parametrized curve $(x(t), y(t))$, the length of the curve on the interval $a \le t \le b$ is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

In the case where $t = \theta$ and $r = f(\theta)$, we have

$$\begin{aligned} x(\theta) &= r\cos\theta = f(\theta)\cos\theta \\ y(\theta) &= r\sin\theta = f(\theta)\sin\theta \end{aligned}$$

and by the product rule,

$$\begin{aligned} \frac{dx}{d\theta} &= f'(\theta)\cos\theta - f(\theta)\sin\theta \\ \frac{dy}{d\theta} &= f'(\theta)\sin\theta + f(\theta)\cos\theta. \end{aligned}$$

After working with the algebra and canceling a few terms, we get that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{dt}\right)^2 = [f'(\theta)]^2 + [f(\theta)]^2,$$

and thus, in polar coordinates, arc length is given by

$$L = \int_a^b \sqrt{[f'(\theta)]^2 + [f(\theta)]^2}\, d\theta.$$

Example 9.4.4. Compute the arc length of the curve $r = 2 - 2\cos\theta$ for $0 \le \theta \le 2\pi$.

We have that $r' = 2\sin\theta$, and thus the arc length integral is

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{[r]^2 + [r']^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{4 - 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta\, d\theta} \\
&= \int_0^{2\pi} \sqrt{8 - 8\cos\theta}\, d\theta \\
&= \int_0^{2\pi} \sqrt{16\sin^2\left(\frac{\theta}{2}\right)}\, d\theta \\
&= \int_0^{2\pi} 4\sin\left(\frac{\theta}{2}\right)\, d\theta \\
&= \left[-8\cos\left(\frac{\theta}{2}\right)\right]_0^{2\pi} \\
&= 16.
\end{aligned}$$

9.4.3 A fun application of polar coordinates

Note: This portion formally relies on some techniques from Calculus III. I will attempt to give intuition for each of the steps and omit formal justification.

Example 9.4.5. Compute $\displaystyle\int_0^\infty e^{-x^2}\, dx$.

We saw before in Example 8.7.10 that it was very difficult to exactly evaluate the error function and we generally had to resort to approximations. However, if we want to evaluate $\operatorname{erf}(x)$ as $x \to \infty$, we can get an exact value with only some cleverness. First we'll suppose that

$$J = \int_0^\infty e^{-x}\, dx$$

is a real number. Since it is real, we can square it (and since x is just a dummy variable, there's no issue with using y for one of the integrals). What's more, since integrals are linear, we can move the constant I inside of the integral.

$$J^2 = J\int_0^\infty e^{-x^2}\, dx = \int_0^\infty e^{-x^2} J\, dx = \int_0^\infty e^{-x^2}\left(\int_0^\infty e^{-y^2}\, dy\right)\, dx.$$

Since e^{-x^2} is not a function of y, it's effectively a constant when integrating with respect to y, so we can move it inside the integral

$$\begin{aligned} J^2 &= \int_0^\infty e^{-x^2} \left(\int_0^\infty e^{-y^2}\, dy \right) dx \\ &= \int_0^\infty \left(\int_0^\infty e^{-x^2} e^{-y^2}\, dy \right) dx \\ &= \int_0^\infty \left(\int_0^\infty e^{-(x^2+y^2)}\, dy \right) dx. \end{aligned}$$

Notice now that we're integrating over all pairs (x, y) where $0 \le x < \infty$ and $0 \le y < \infty$. This is exactly the first quadrant (QI) in the Cartesian plane. In polar coordinates, this plane is described by all pairs (r, θ) where $0 \le r < \infty$ and $0 \le \theta \le \frac{\pi}{2}$. Since $e^{-x^2-y^2} = e^{-r^2}$, it seems reasonable that we might want to change to polar coordinates.

The main technical issue is then figuring out how to replace dx and dy correctly so that we can work with dr and $d\theta$. As it turns out, $dx\, dy = r\, dr\, d\theta$, and the idea behind it is this: an infinitesimal rectangle in the plane with sides dx and dy has infinitesimal area $dA = dx \cdot dy$, and if you try to describe this same area in polar coordinates, you end up getting that $dA = r \cdot dr \cdot d\theta$. Thus

$$\begin{aligned} J^2 &= \int_0^\infty \left(\int_0^\infty e^{-(x^2+y^2)}\, dy \right) dx \\ &= \int_0^{\pi/2} \left(\int_0^\infty e^{-r^2} r\, dr \right) d\theta \\ &= \int_0^{\pi/2} \left[-\frac{1}{2} e^{-r^2} \right]_0^\infty d\theta \\ &= \int_0^{\pi/2} \frac{1}{2}\, d\theta \\ &= \left[\frac{1}{2}\theta \right]_0^{\pi/2} \\ &= \frac{\pi}{4} \\ \Rightarrow I &= \frac{\sqrt{\pi}}{2}. \end{aligned}$$