MAT265 Calculus for Engineering I

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Contents

0 Prerequisites

0.1 Review of Functions

0.1.1 The Definition

Definition. A *function* f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B. We write $f : A \to B$ to formally represent the above.

The set A above is called the *domain* and the set B is called the *codomain*, if **every** element in B can be written as $f(x)$ for some x, then we call B the *range*. There are many important terms associated with functions:

- **Independent Variable**: associated with the domain of a function, i.e. the x variable.
- **Dependent Variable**: associated with the range of a function, i.e. the $f(x)$'s.
- **Graph of a Function**: the set of all points of the form $(x, f(x))$ where x varies throughout the entire domain.
- **Argument of a Function**: the expression on which the function is evaluated.

  For example: x is the argument of $f(x)$; 7 is the argument of $f(7)$; $x^5 - 45$ is the argument of $f(x^5 - 45)$

0.1.2 Catalog of Essential Functions

1. **Polynomials**: are functions of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

   - The a_i's are called the *coefficients* of the polynomial.
   - The number n is called the *degree* of the polynomial.

2. **Rational Functions**: are functions of the form $\dfrac{p(x)}{q(x)}$ where p and q are functions

   For example: $\dfrac{5x^3 - 13}{2x^2 - x + 5}$

3. **Algebraic Functions**: are functions constructed using algebraic operations

   For example: $f(x) = \sqrt{x^5 - 7x + 5}$; $g(x) = x^{1/7}(x^2 - 2)$

4. **Exponential Functions**: have the form $f(x) = b^a$, where $b \neq 1$ is a positive real number. Logarithmic functions go hand-in-hand with these. For the following important rules of exponential and logarithmic functions, let $b \neq 1$ be a positive real number.

   - $b^x b^y = b^{x+y}$ for all real numbers x and y
   - $(b^x)^y = b^{xy}$
   - $\log_b(xy) = \log_b(x) + \log_b(y)$ for all positive x and y
   - $\log_b(x^y) = y \log_b(x)$ for all real numbers y and positive real numbers x

5. **Trigonometric Functions**: $\sin(x), \cos(x), \tan(x)$ and so on. These are fundamental to many branches of mathematics and engineering.

6. **Piece-wise Functions**: As the name suggests, these are functions comprised of pieces of other functions. For example:

$$f(x) = \begin{cases} \sin(x) & \text{if } x < 0 \\ x & \text{if } 0 \le x \le 1 \\ e^{x-1} & \text{if } x > 1 \end{cases}$$

0.1.3 Transformations of Functions

Shifts/Translations: let $c > 0$

1. $f(x) + c$ shifts the function f up by c
2. $f(x) - c$ shifts the function f down by c
3. $f(x + c)$ shifts the function f to the left by c
4. $f(x - c)$ shifts the function f to the right by c

Stretches and Reflections: let $c > 1$

1. $cf(x)$ stretches f vertically by a factor of c
2. $\frac{1}{c}f(x)$ compresses f vertically by a factor of c
3. $f(cx)$ compresses f horizontally by a factor of c
4. $f(\frac{1}{c}x)$ stretches f horizontally by a factor of c
5. $-f(x)$ reflects f about the x-axis
6. $f(-x)$ reflects f about the y-axis

Combinations of Functions:

1. $(f + g)(x) = f(x) + g(x)$
2. $(f - g)(x) = f(x) - g(x)$
3. $(f \cdot g)(x) = f(x) \cdot g(x)$
4. $(\dfrac{f}{g})(x) = \dfrac{f(x)}{g(x)}$, on the proper domain
5. ***IMPORTANT!*** $(f \circ g)(x) = f(g(x))$, a composition of functions

0.1.4 Inverse Functions

Definition. Let $f : A \to B$ be a function. If there exists a function $g : B \to A$ so that $f \circ g : B \to B$ is the identity function for B **and** $g \circ f : A \to A$ is the identity function for A, we call g the inverse function of f and denote it f^{-1}.

There may not always be an inverse function for any given f. This brings up the need for the following definitions.

Definition. A function $f : A \to B$ is called *one-to-one* if for every element b in the set B, there is **at most** one element a in the set A such that $f(a) = b$.

Definition. A function $f : A \to B$ is called *onto* if for every element b in the set B, there is **at least** one element a in the set A such that $f(a) = b$.

Proposition 0.1.1. *A function $f : A \to B$ has an inverse function if and only if f is both one-to-one and onto.*

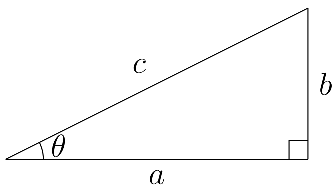
If a function does not have an inverse, not all is lost. The trick is to find an interval where f is both one-to-one and onto, then just pretend that the restricted domain and range were the original ones.

Basic idea for finding inverses of a function f:

1. Find an interval where f is one-to-one and onto.
2. Replace $f(x)$ with a simpler symbol (might I suggest the letter y?).
3. Switch the roles of x and y in the equation.
4. Solve the above equation for y.
5. Replace the symbol y with $f^{-1}(x)$.

0.2 Trigonometric Identities

Trigonometric Functions



From the right triangle pictured above, we have the following function definitions

$$\sin(\theta) = \frac{b}{c} \qquad \cos(\theta) = \frac{a}{c} \qquad \tan(\theta) = \frac{b}{a}$$

$$\csc(\theta) = \frac{c}{b} \qquad \sec(\theta) = \frac{c}{a} \qquad \cot(\theta) = \frac{a}{b}$$

Angle Sum/Difference Formulas

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Double-Angle Formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Power Reducing Formulas

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$
$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$
$$\tan^2\theta = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$$

Half-Angle Formulas

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos(\theta)}{2}}$$
$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 + \cos(\theta)}{2}}$$
$$\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos(\theta)}{1 + \cos(\theta)}}$$

Product-to-Sum Formulas

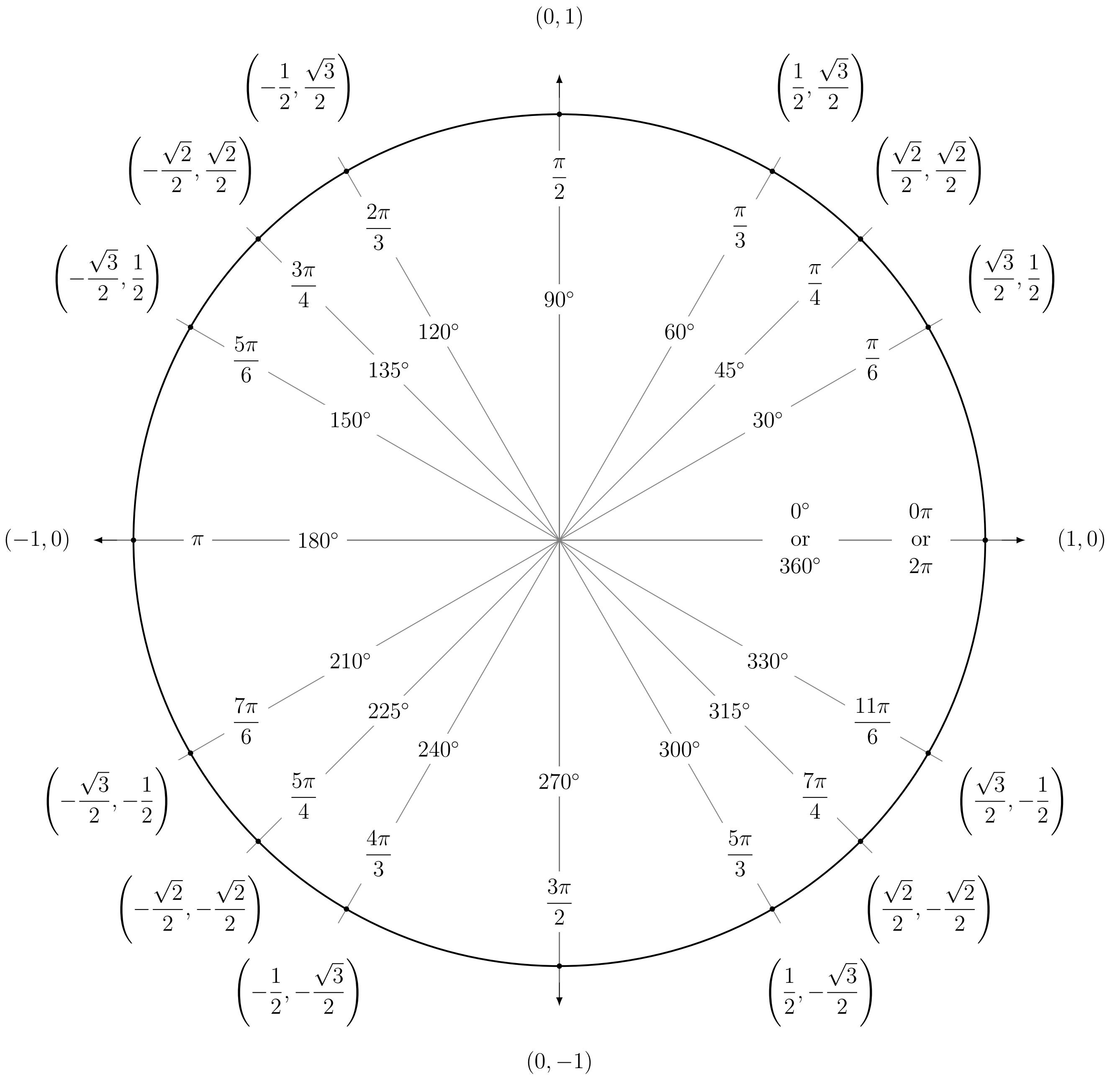
$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right]$$
$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha - \beta) + \cos(\alpha + \beta)\right]$$
$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha + \beta) + \sin(\alpha - \beta)\right]$$
$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha + \beta) - \sin(\alpha - \beta)\right]$$

Sum-to-Product Formulas

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$
$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha - \beta}{2}\right)\cos\left(\frac{\alpha + \beta}{2}\right)$$
$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$
$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$$

0.2.1 The Unit Circle

Points on the unit circle are given by $(x, y) = (\cos\theta, \sin\theta)$. The most important angles to know are listed below, along with the relevant coordinates on the unit circle. To remember this most efficiently, it really suffices just to remember the first quadrant as there is plenty of symmetry.



1 Functions and Limits

1.3 The Limit of a Function

Definition. Let f be a function and suppose that $f(x)$ is defined for all x very near the number a. If we can pick x-values so that $f(x)$ is arbitrarily close to some number L when x sufficiently close to a (when both $x < a$ and $x > a$), then we write

$$\lim_{x \to a} f(x) = L$$

and say "the **limit** of $f(x)$, as x approaches a, is L."

Remark. Notice that the definition does not require that $f(x)$ be defined when $x = a$. Notice that the limit also requires that we can approach from either side of a to get the same L value.

Example 1.3.1. Using a table of values, guess the limit of the function $f(x) = \dfrac{x^2 - 16}{x + 4}$ as $x \to -4$.

$x < -4$	$f(x)$
-4.1	-8.1
-4.01	-8.01
-4.001	-8.001
-4.0001	-8.0001
-4.00001	-8.00001

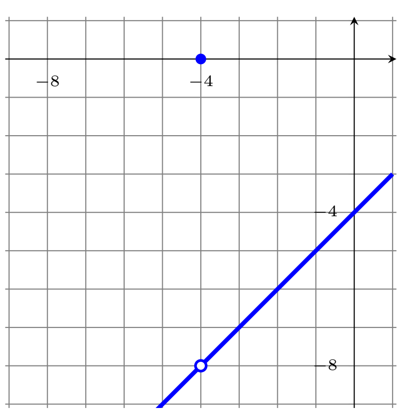
$x > -4$	$f(x)$
-3.9	-7.9
-3.99	-7.99
-3.999	-7.999
-3.9999	-7.9999
-3.99999	-7.99999

Limit: -8

Example 1.3.2. Let g be the function given by

$$g(x) = \begin{cases} \dfrac{x^2 - 16}{x + 4} & \text{if } x \neq -4, \\ 0 & \text{if } x = -4. \end{cases}$$

Use a graph to determine $\lim_{x \to -4} g(x)$.



Limit: -8

Remark. This last example demonstrates that, even if $f(a)$ is defined, $\lim_{x\to a} f(x)$ is independent of the function's value at a.

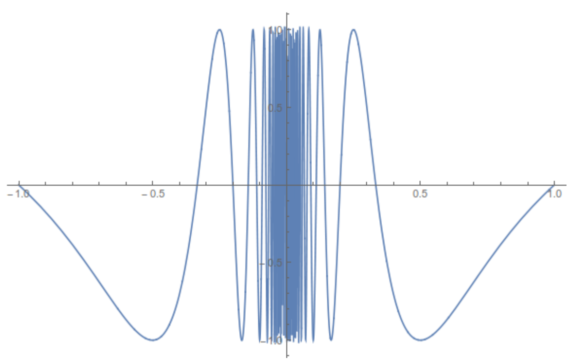
Example 1.3.3. Consider the function $f(x) = \cos\left(\frac{\pi}{2x}\right)$. Using a table of values, guess the limit $\lim_{x\to 0} f(x)$.

$x < 0$	$f(x)$
-0.05	1
-0.01	1
-0.005	1
-0.001	1
-0.0005	1

$x > 0$	$f(x)$
0.05	1
0.01	1
0.005	1
0.001	1
0.0005	1

Limit: 1?

Now look at the graph of f below. Notice that the output doesn't just settle on 1, but rather oscillates rapidly at $x \to 0$. Since $f(x)$ never actually settles on a number at all (it hits every value between -1 and 1 infinitely many times), we have that the limit does not exist. This is one of the pitfalls that we can run into if we just use the calculator to guess and check what limits may be.



Example 1.3.4. Using a table of values, determine the limit $\lim_{x\to 2} \frac{1}{(x-2)^2}$, if it exists.

$x < 2$	$f(x)$
1	1
1.9	100
1.99	10^4
1.999	10^6
1.9999	10^8

$x > 2$	$f(x)$
3	1
2.1	100
2.01	10^4
2.001	10^4
2.0001	10^8

Limit: Does Not Exist

Remark. ∞ is *not* a real number, so the above limit *does not exist*. However, we will discuss these situations more in a future lecture.

Definition. Let f be a function and suppose that $f(x)$ is defined for all x very near the number a with $x < a$. If we can pick x-values so that $f(x)$ is arbitrarily close to some number L when x sufficiently close to a, then we write

$$\lim_{x \to a^-} f(x) = L$$

and say "the **limit** of $f(x)$, as x approaches a **from the left**, is L."

Similarly, if instead requiring that $x > a$, we write

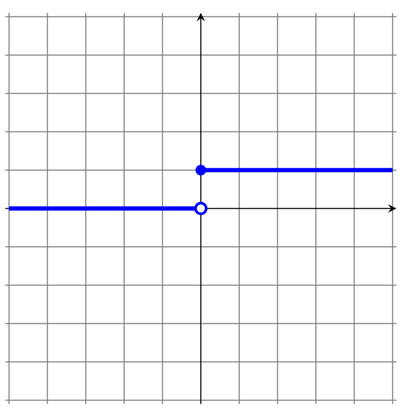
$$\lim_{x \to a^+} f(x) = L$$

and say "the **limit** of $f(x)$, as x approaches a **from the right**, is L."

Example 1.3.5. The *Heaviside function* is the function is given by

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 1. \end{cases}$$

Use a graph to determine the limit of $H(x)$ as $x \to 0$.



Limit: Does Not Exist

The Heaviside function helps to motivate the following result.

Proposition 1.3.6. *Given a function f and a real number a,*

$$\lim_{x \to a} f(x) = L \text{ if and only if both } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$

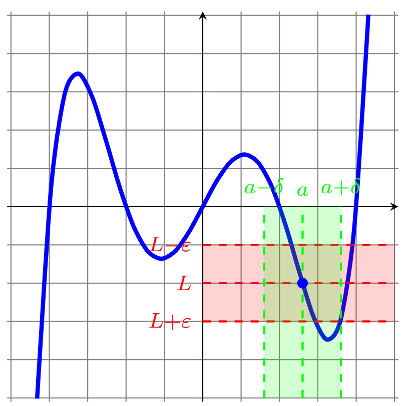
You will not be expected to use it or memorize it in this course, but I think it's good to have seen the formal definition of a limit. As I mentioned on the first day - this definition right here came 150 years after calculus had been invented and is what allowed mathematicians to make Newton's and Leibniz's ideas rigorous.

Definition. Let f be a function defined on an open interval containing a, except possibly at a itself, and let L be a real number. Then we say

$$\lim_{x \to a} f(x) = L$$

if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

In words, this definition says that, given any small vertical ε-sized window, we can find some horizontal δ-sized window containing the line $x = a$ so that the entire graph of the function of f in the horizontal window *also* sits inside the vertical window. Pictorially,



Thinking briefly about the graph of $f(x) = \dfrac{1}{(x-2)^2}$, with this definition it's clear why the function has no limit as $x \to 2$ - no matter what vertical window we set, there's always some part of the graph of f near 2 that will sit outside of that vertical window.

1.4 Calculating Limits

Theorem 1.4.1 (Algebraic Laws of Limits)**.** *Let c be a constant and let f and g be functions such that the limits*

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)$$

exist. We have the following algebraic rules for limits:

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

3. $\lim_{x\to a}[cf(x)] = c\left[\lim_{x\to a} f(x)\right]$

4. $\lim_{x\to a}[f(x)\cdot g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right]$

5. *If* $\lim_{x\to a} g(x) \neq 0$*, then* $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

6. *If n is a positive integer, then* $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$

7. $\lim_{x\to a} c = c$

8. $\lim_{x\to a} x = a$

9. *If n is a positive integer, then* $\lim_{x\to a} x^n = a^n$

10. *If n is a positive integer, then* $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$

    [If n is even, we assume $a > 0$.]

11. *If n is a positive integer, then* $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$

    [If n is even, we assume $\lim_{x\to a} f(x) > 0$*.]*

Example 1.4.2. Evaluate the following limit and justify each step: $\lim_{x\to 2}(4x^2+3)$.

$$\begin{aligned}
\lim_{x\to 2}(4x^2+3) &= \lim_{x\to 2} 4x^2 + \lim_{x\to 2} 3 && \text{(By \# 1)}\\
&= 4\left[\lim_{x\to 2} x^2\right] + \lim_{x\to 2} 3 && \text{(By \# 3)}\\
&= 4\left[\lim_{x\to 2} x^2\right] + 3 && \text{(By \# 7)}\\
&= 4(2)^2 + 3 && \text{(By \# 9)}\\
&= 19
\end{aligned}$$

Example 1.4.3. Evaluate the following limit and justify each step: $\lim_{x\to 2} \frac{x^2+x+2}{x+1}$.

$$\begin{aligned}
\lim_{x\to 2} \frac{x^2+x+2}{x+1} &= \frac{\lim_{x\to 2}(x^2+x+2)}{\lim_{x\to 2}(x+1)} && \text{(By \# 5)} \\
&= \frac{\lim_{x\to 2} x^2 + \lim_{x\to 2} x + \lim_{x\to 2} 2)}{\lim_{x\to 2} x + \lim_{x\to 2} 1} && \text{(By \# 1)} \\
&= \frac{\lim_{x\to 2} x^2 + \lim_{x\to 2} x + 2)}{\lim_{x\to 2} x + 1} && \text{(By \# 7)} \\
&= \frac{(2)^2 + (2) + 2)}{(2)+1} && \text{(By \# 9)} \\
&= \frac{8}{3}.
\end{aligned}$$

Proposition 1.4.4 (Direct Substitution Property)**.** *If f is a polynomial or rational function and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$.*

Remark. The trigonometric functions also satisfy this property, as do exponential functions. As we'll see, there are many general types of functions with this property.

Example 1.4.5. Find $\lim_{x\to -4} f(x)$ where $f(x) = \frac{x^2-16}{x+4}$.

Note that for limits, we only need x to be arbitrarily close to -4 and not actually equal to it. This means that $x \neq 4$, i.e., that $x+4 \neq 0$.

$$\begin{aligned}
\lim_{x\to -4} \frac{x^2-16}{x+4} &= \lim_{x\to -4} \frac{(x+4)(x-4)}{x+4} \\
&= \lim_{x\to -4} (x-4) \\
&= 8.
\end{aligned}$$

This shows us that f behaves exactly like the function $g(x) = x-4$ everywhere except at $x = 4$.

Proposition 1.4.6. *If $f(x) = g(x)$ when $x \neq a$, then, provided the limit exists, $\lim_{x\to a} f(x) = g(x)$*

Example 1.4.7. Find $\lim_{t\to 0} \frac{\sqrt{t+1}-1}{t}$.

Once again, with limits, we only care that t get arbitrarily close to 0. Since $t \neq 0$,

$$\begin{aligned}
\lim_{t\to 0} \frac{\sqrt{t+1}-1}{t} &= \lim_{t\to 0} \frac{\sqrt{t+1}-1}{t}\left(\frac{\sqrt{t+1}+1}{\sqrt{t+1}+1}\right) \\
&= \lim_{t\to 0} \frac{(t+1)-1}{t(\sqrt{t+1}+1)} \\
&= \lim_{t\to 0} \frac{t}{t(\sqrt{t+1}+1)} \\
&= \lim_{t\to 0} \frac{1}{\sqrt{t+1}+1} \\
&= \frac{1}{\sqrt{0+1}+1} \\
&= \frac{1}{2}.
\end{aligned}$$

Example 1.4.8. Find $\lim_{t\to 0} |t|$. First recall that

$$|t| = \begin{cases} t & \text{if } t \geq 0 \\ -t & \text{if } t < 0. \end{cases}$$

We have to use this piecewise definition of $|t|$ and take limits from the left and right, because we don't have a rule that says what to do with absolute values.

$$\begin{aligned}
\lim_{t\to 0^-} |t| &= \lim_{t\to 0^-} -t && \text{(since } t < 0\text{)} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to 0^+} |t| &= \lim_{t\to 0^+} t && \text{(since } t > 0\text{)} \\
&= 0
\end{aligned}$$

since $\lim_{t\to 0^-} |t| = \lim_{t\to 0^+} |t| = 0$, we must have that

$$\lim_{t\to 0} |t| = 0.$$

Theorem 1.4.9. *If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then*

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$$

Theorem 1.4.10 (The Squeeze Theorem*)**.** *If $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then*

$$\lim_{x\to a} g(x) = L.$$

Example 1.4.11. Show that $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

First recall that the range of $\sin(t)$ is $[-1, 1]$, so $-1 \le \sin(t) \le 1$ for all t. So

$$\begin{aligned} -1 &\le \sin(x) \le 1 \\ -x^2 &\le x^2 \sin(x) \le x^2 \qquad\qquad (\text{since } x^2 \ge 0) \end{aligned}$$

Since $\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$, by the Squeeze theorem, we have that

$$\lim_{x\to 0} x^2 \sin(x) = 0.$$

The following result uses the Squeeze Theorem, but the proof is a bit geometric and round-about (although you can read it in the book). Instead, we'll accept it as fact for now and we will provide a "slicker" proof later.

Fact. $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$

Example 1.4.12. Find $\lim_{x\to 0} \frac{\tan(x)}{x}$.

Recall that $\tan(x) = \frac{\sin(x)}{\cos(x)}$. Then

$$\begin{aligned} \lim_{x\to 0} \frac{\tan(x)}{x} &= \lim_{x\to 0} \frac{\sin(x)}{x\cos(x)} \\ &= \lim_{x\to 0} \left(\frac{\sin(x)}{x}\right)\left(\frac{1}{\cos(x)}\right) \\ &= \left(\lim_{x\to 0} \frac{\sin(x)}{x}\right)\left(\lim_{x\to 0} \frac{1}{\cos(x)}\right) \\ &= (1)(1) \\ &= 1 \end{aligned}$$

*In some languages, like German and Russian, the Squeeze Theorem is also known as the *Two Policemen (and a Drunk) Theorem.*

Example 1.4.13. Find $\lim_{x \to 0} \dfrac{3\sin(4x)}{5x}$.

$$\begin{aligned}
\lim_{x \to 0} \frac{3\sin(4x)}{5x} &= \lim_{x \to 0} \frac{3\sin(4x)}{5x}\left(\frac{4}{4}\right) \\
&= \lim_{x \to 0} \frac{12}{5}\left(\frac{\sin(4x)}{4x}\right) \\
&= \frac{12}{5}\left(\lim_{x \to 0} \frac{\sin(4x)}{4x}\right)
\end{aligned}$$

We make the substitution $t = 4x$. Then as $x \to 0$, $t \to 0$, so we get

$$\begin{aligned}
&= \frac{12}{5}\left(\lim_{t \to 0} \frac{\sin(t)}{t}\right) \\
&= \frac{12}{5}(1) \\
&= \frac{12}{5}.
\end{aligned}$$

1.5 Continuity

Definition. A function f is **continuous at** a if $\lim_{x\to a} f(x) = f(a)$.

Remark. The definition above implicitly requires the following three things if f is continuous at a:

- $\lim_{x\to a} f(x)$ exists
- $f(a)$ exists
- The two values agree.

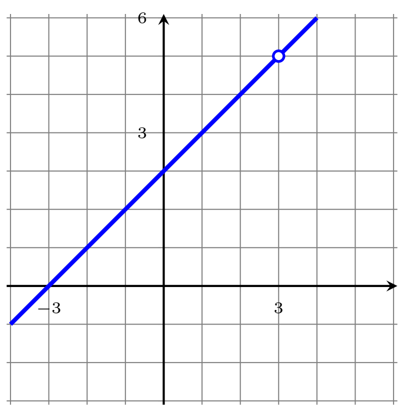
Intuitively, it means that we can draw the graph of f (near a) without having to lift the pencil off of the page.

Definition. If $f(x)$ is defined for all x near a but f is not continuous at a, we say that f is **discontinuous at** a, or alternatively that f has a **discontinuity at** a.

All discontinuities are not created equal.

Example 1.5.1. Where is the following function discontinuous? Graph the function.

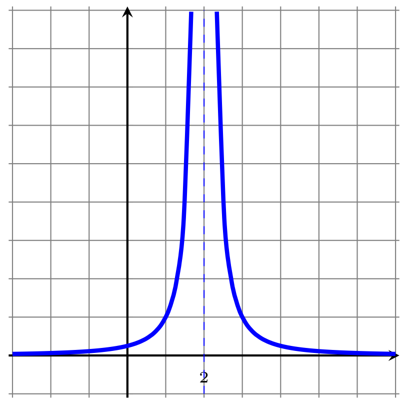
$$f(t) = \frac{t^2 - 9}{t - 3}$$



Discontinuity: $\underline{t = 6}$

Example 1.5.2. Where is the following function discontinuous? Graph the function.

$$f(x) = \frac{1}{(x-2)^2}$$

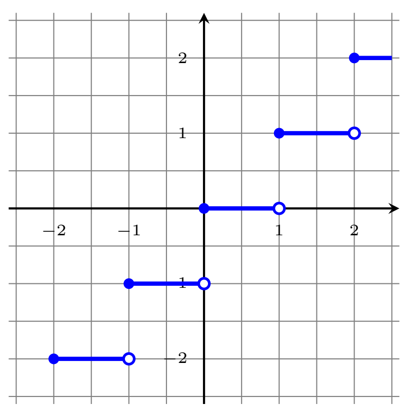


Discontinuity: $\underline{x = 2}$

Example 1.5.3. Where is the following function discontinuous? Graph the function.

$$f(x) = \lfloor x \rfloor$$

This function is called the "floor function". It rounds a number down to the nearest integer.



Discontinuities:
$x = n$, for every integer n

Definition. Let f be a function that is discontinuous at a.

We say that a is a **removable discontinuity** if we can find a function g such that g is continuous at a and $f(x) = g(x)$ for all $x \neq a$. [See Example 1.5.1]

We say that a is an **infinite discontinuity** if $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$ are unbounded (i.e. "go to $\pm\infty$"). [See Example 1.5.2]

We say that a is a **jump discontinuity** if $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$ when both one-sided limits exist. [See Example 1.5.3]

Theorem 1.5.4 (Algebraic Laws of Continuous Functions)**.** *Let c be some constant and let f and g be functions. Suppose that f and g are continuous at a. Then each of the following functions are continuous at a:*

1. *$f+g$, where $(f+g)(x) = f(x) + g(x)$*
2. *$f-g$, where $(f-g)(x) = f(x) - g(x)$*
3. *cf, where $(cf)(x) = cf(x)$*
4. *fg, where $(fg)(x) = f(x)g(x)$*
5. *$\dfrac{f}{g}$ if $g(a) \neq 0$, where $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$*

We will prove that fg is continuous at a, but the rest all follow similarly and are left as an exercise to the reader.

Proof. Since f and g are both continuous at a, we have that

$$\lim_{x\to a} f(x) = f(a) \quad \text{and} \quad \lim_{x\to a} g(x) = g(a).$$

Since these limits exist, we can apply our algebraic laws of limits to get

$$\begin{aligned} \lim_{x\to a}(fg)(x) &= \lim_{x\to a}\left[f(x)g(x)\right] \\ &= \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right] \\ &= f(a)g(a) \\ &= (fg)(a), \end{aligned}$$

and so fg is continuous at a.

□

Proposition 1.5.5. *Every polynomial* $p(x) = c_n x^n + \cdots + c_1 x + c_0$ *is continuous on* $(-\infty, \infty)$.

Proof. Certainly

$$p(a) = c_n a^n + \cdots + c_1 a + c_0$$

is defined for every real number a. So, using our algebraic limit laws, we get

$$\begin{aligned}
\lim_{x\to a} p(x) &= \lim_{x\to a} (c_n x^n + \cdots + c_1 x + c_0) \\
&= \lim_{x\to a} (c_n x^n) + \cdots + \lim_{x\to a} (c_1 x) + \lim_{x\to a} c_0 \\
&= c_n \left(\lim_{x\to a} x^n\right) + \cdots + c_1 \left(\lim_{x\to a} x\right) + \lim_{x\to a} c_0 \\
&= c_n a^n + \cdots + c_1 a + c_0 \\
&= p(a),
\end{aligned}$$

and therefore $p(x)$ is continuous at every real number a.

□

Corollary 1.5.6. *Every rational function* $f(x) = \dfrac{p(x)}{q(x)}$, *where* p *and* q *are polynomials, is continuous on its domain.*

Fact. Root functions, trigonometric functions, exponential functions, and logarithmic functions are all continuous on their domains.

The following theorem demonstrates an important fact about the interplay between limits and continuous functions.

Theorem 1.5.7. *If* f *is continuous at* b *and* $\lim_{x\to a} g(x) = b$, *then* $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$

Proof. See Appendix D. □

Proposition 1.5.8. *If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.*

Proof. Since g is continuous at a,

$$\lim_{x \to a} g(x) = g(a).$$

Since f is continuous at $g(a)$, then by Theorem 1.5.7

$$\begin{aligned}
\lim_{x \to a} (f \circ g)(x) &= \lim_{x \to a} f(g(x)) \\
&= f\left(\lim_{x \to a} g(x)\right) \\
&= f(g(a)) \\
&= (f \circ g)(a),
\end{aligned}$$

and therefore $f \circ g$ is continuous at a.

□

Definition. Let f be a function. f is **continuous from the left at** a if $\lim_{x \to a^-} f(x) = f(a)$ and is **continuous from the right at** a if $\lim_{x \to a^+} f(x) = f(a)$. f is **continuous on an interval** if it is continuous at every point in that interval (here we assume that it is only continuous from the left/right at the endpoints of the interval, if they are included).

Remark. If we say f is **continuous** without any further qualification, *in this class* we will assume that f is continuous on $(-\infty, \infty)$.

Example 1.5.9. The floor function from Example 1.5.3 is continuous from the right, but not from the left.

To see this, let $a = n$ for any integer n. We then have that

$$\lim_{x \to n^-} \lfloor x \rfloor = n - 1 \neq \lfloor n \rfloor,$$

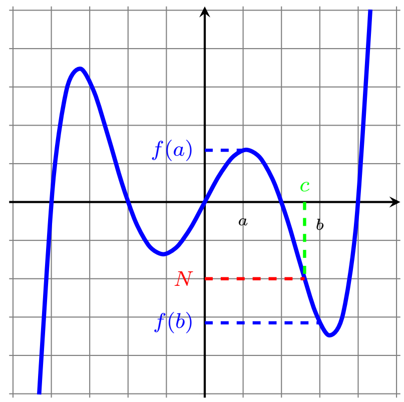
so at each n, $\lfloor x \rfloor$ is not continuous from the left. However,

$$\lim_{x \to n^+} \lfloor x \rfloor = n = \lfloor n \rfloor,$$

so at each n, $\lfloor x \rfloor$ is continuous from the right.

Theorem 1.5.10 (Intermediate Value Theorem)**.** *Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in the open interval (a, b) such that $f(c) = N$.*

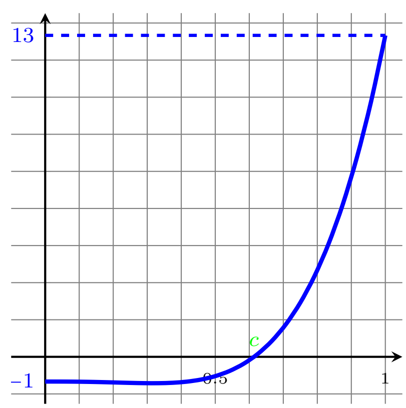
Although the proof of this theorem is beyond the scope of the course, we can demonstrate with a graph below why it intuitively makes sense.



Another way to think about it is to ask the following question (incredulously). "If f is continuous, how can we possibly draw its graph *without* crossing the line $y = N$?"

Example 1.5.11. Show that the polynomial $p(x) = 5x^5 + 17x^4 - 8x^3 - 1$ has a root between 0 and 1.

Recall that a "root" of a polynomial is a number c such that $p(c) = 0$, and corresponds to an x-intercept of the graph of p.



Since $p(x)$ is continuous, $p(0) = -1$, and $p(1) = 13$, by the Intermediate Value Theorem there must exist some real number $0 < c < 1$ such that $p(c) = 0$.

1.6 Limits Involving Infinity

We saw previously that $\lim_{x\to 2} \frac{1}{(x-2)^2}$ did not exist, as the function grew unbounded as $x \to 2$. Similarly, $\lim_{x\to -2} \frac{-1}{(x-2)^2}$ does not exist, but the graph is different - rather than growing positively unbounded, this functions becomes negatively unbounded. Just saying that a limit "does not exist" does not really capture the behavior of the graph. So, to emphasize the difference, we write

$$\lim_{x\to 2} \frac{1}{(x-2)^2} = \infty \quad \text{and} \quad \lim_{x\to 2} \frac{-1}{(x-2)^2} = -\infty.$$

Now, ∞ is *not a real number* so we're not contradicting our statement that the limit does not exist. This is just a slight abuse of notation.

Remark. This same notation can be used for limits from the left and limits from the right also.

Definition. The vertical line $x = a$ is called a **vertical asymptote** for the curve $y = f(x)$ if at least one of the following is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$
$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty.$$

Example 1.6.1. Let $f(t) = \frac{1}{t-1}$. Find $\lim_{t\to 1^-} f(t)$ and $\lim_{t\to 1^+} f(t)$. List any vertical asymptotes of the graph of $f(t)$.

Notice that $f(t) < 0$ for $t < 1$, and since the denominator is getting smaller and smaller as t approaches 1 from the left, we have that

$$\lim_{t\to 1^-} f(t) = -\infty.$$

Similarly, $f(t) > 0$ for $t > 1$, and since the denominator is getting smaller and smaller as t approaches 1 from the right, we have that

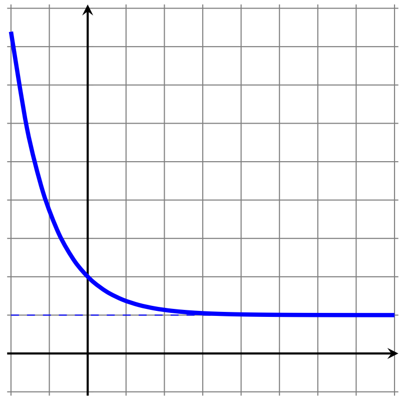
$$\lim_{t\to 1^+} f(t) = \infty.$$

Indeed, $f(t)$ has only a single discontinuity at $t = 1$, so $x = 1$ is the only vertical asymptote.

Example 1.6.2. Determine all vertical asymptotes of the curve $y = \csc(x)$. Recall that $\csc(x) = \frac{1}{\sin(x)}$. Recall also that $\sin(x) = 0$ when $x = n\pi$ for any integer n. The behavior of the left and right limits is different depending on whether n is even or odd.

Notice that $\csc(x)$ is undefined whenever $\sin(x)$, i.e., whenever $x = n\pi$ for any integers n. By a similar argument as in Example 1.6.1, we see that we have vertical asymptotes at $x = n\pi$, for all integers n.

Example 1.6.3. Graph the function $f(x) = e^{-x} + 1$. What do you notice about about $f(x)$ as x gets arbitrarily large?



As x becomes larger and larger, $f(x)$ gets closer and closer to 1. Indeed, we can get arbitrarily close to 1 by choosing sufficiently large x. So, we write

$$\lim_{x\to\infty} f(x) = 1.$$

Remark. Because all real numbers are less than infinity, we can only ever really "approach ∞" from the left. Similarly, we can only ever really "approach $-\infty$" from the right. We do not use the "from the left" or "from the right" limit notation when looking at $x \to \pm\infty$.

Definition. The horizontal line $y = L$ is a **horizontal asymptote** for the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.$$

Example 1.6.4. Find $\displaystyle\lim_{x\to\infty} \frac{1}{x}$. Find $\displaystyle\lim_{x\to-\infty} \frac{1}{x}$.

Notice that as x tends toward ∞, $\frac{1}{x}$ becomes positively smaller and smaller, hence

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

Similarly, as x tends toward $-\infty$, $\frac{1}{x}$ becomes negatively smaller and smaller, hence

$$\lim_{x\to-\infty} \frac{1}{x} = 0.$$

Proposition 1.6.5.

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \quad \textit{and} \quad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

Proof. The proof follows from the previous example and the product of limits property. □

Example 1.6.6. Find any horizontal asymptotes for the function $f(x) = \dfrac{8x^3+2x+1}{16x^3-147}$.

To find the horizontal asymptotes, we must take limits as $x \to \pm\infty$.

$$\begin{aligned}
\lim_{x\to-\infty} \frac{8x^3+2x+1}{16x^3-147} &= \lim_{x\to-\infty} \frac{x^3\left(8+\frac{2}{x^2}+\frac{1}{x^3}\right)}{x^3\left(16-\frac{147}{x^3}\right)} \\
&= \lim_{x\to-\infty} \frac{8+\frac{2}{x^2}+\frac{1}{x^3}}{16-\frac{147}{x^3}} \\
&= \frac{8+0+0}{16-0} \\
&= \frac{8}{16} = \frac{1}{2}.
\end{aligned}$$

By a similar argument, we have $\lim_{x\to\infty} f(x) = \dfrac{1}{2}$, hence we have a single horizontal asymptote $y = \frac{1}{2}$.

The technique applied in this example proves the following proposition.

Proposition 1.6.7. *Consider the rational function*

$$f(x) = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0},$$

where m, n are positive integers, $a_0, \ldots, a_n, b_0, \ldots, b_m$ are real numbers, and a_n, b_m are nonzero.

- *If $n > m$, then f has no horizontal asymptotes.*
- *If $n = m$, then f has one horizontal asymptote: $y = \frac{a_n}{b_m}$.*
- *If $n < m$, then f has one horizontal asymptote: $y = 0$.*

Example 1.6.8. Find all horizontal asymptotes of the function $f(x) = \dfrac{\sqrt{3x^2+7x+1}}{5x-11}$.

We'll use an alternative definition of the absolute value, $|x| = \sqrt{x^2}$, then appeal to the piecewise definition.

$$\begin{aligned}
\lim_{x\to-\infty} \frac{\sqrt{3x^2+7x+1}}{5x-11} &= \lim_{x\to-\infty} \frac{\sqrt{x^2\left(3+\frac{7}{x}+\frac{1}{x^2}\right)}}{x\left(5-\frac{11}{x}\right)} \\
&= \lim_{x\to-\infty} \frac{|x|\sqrt{3+\frac{7}{x}+\frac{1}{x^2}}}{x\left(5-\frac{11}{x}\right)} \\
&= \lim_{x\to-\infty} -\frac{\sqrt{3+\frac{7}{x}+\frac{1}{x^2}}}{5-\frac{11}{x}} \qquad \text{(since } |x| = -x \text{ for } x < 0\text{)} \\
&= \lim_{x\to-\infty} -\frac{\sqrt{3+0+0}}{5-0} \\
&= -\frac{\sqrt{3}}{5},
\end{aligned}$$

and a similar argument shows us that $\lim_{x\to\infty} \dfrac{\sqrt{3x^2+7x+1}}{5x-11} = \dfrac{\sqrt{3}}{5}$. So we have two horizontal asymptotes: $x = -\frac{\sqrt{3}}{5}$ and $x = \frac{\sqrt{3}}{5}$.

Example 1.6.9. Find $\lim_{x\to\infty} \cos\left(\frac{1}{x}\right)$.

By Proposition 1.6.5, we have that

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

and so by Theorem 1.5.7, we have that

$$\begin{aligned}\lim_{x\to\infty} \cos\left(\frac{1}{x}\right) &= \cos\left(\lim_{x\to\infty} \frac{1}{x}\right)\\ &= \cos(0)\\ &= 1.\end{aligned}$$

Example 1.6.10. Find $\lim_{x\to\infty} \cos(x)$ if it exists.

Notice that as x tends toward ∞, $\cos(x)$ achieves every value in the interval $[-1, 1]$ infinitely many times. As such, $\cos(x)$ never tends toward any single real number and the limit does not exist.

Given a function f, the following notation indicates that the value $f(x)$ grows without bound (positively or negatively) as $x \to \infty$ or $x \to -\infty$:

$$\lim_{x\to\infty} f(x) = \infty \qquad \lim_{x\to-\infty} f(x) = \infty$$

$$\lim_{x\to\infty} f(x) = -\infty \qquad \lim_{x\to-\infty} f(x) = -\infty$$

Again, we're *not* saying that ∞ is a number or that $f(x)$ has a horizontal asymptote $y = \infty$. This notation is merely suggestive of the behavior of the graph.

Remark. Although we're throwing around ∞ all over the place, we have to remember to exercise caution; we cannot treat it like a real number and/or blindly apply our algebraic limit laws as though it were.

Example 1.6.11. Consider $f(x) = x^3 - x^2$, and suppose we want to find $\lim_{x\to\infty} f(x)$. If we could apply the algebraic laws of limits, we would have

$$\lim_{x\to\infty} x^3 - x^2 = \lim_{x\to\infty} x^3 - \lim_{x\to\infty} x^2 = \infty - \infty.$$

But "$\infty - \infty$" is undefined. You may be tempted to make it 0, but then this would disagree with the graph of $f(x)$. Instead, we can write

$$\lim_{x\to\infty} x^3 - x^2 = \lim_{x\to\infty} x^2(x-1) = \infty$$

as both x^2 and $x - 1$ grow arbitrarily large as $x \to \infty$.

2 Derivatives

2.1 Derivatives and Rates of Change

Definition. Given a function f defined on an interval $[a, b]$, the **average rate of change from $x = a$ to $x = b$** is

$$\frac{f(a) - f(b)}{a - b}.$$

Remark. The average rate of change is the slope of the line connecting the two points $(a, f(a))$ and $(b, f(b))$.

Example 2.1.1. The function $s(t) = -16t^2 + 32t$, $[0, 2]$, represents the vertical height in feet (as a function of time) of a dropped rubber ball during one bounce. What is the average velocity of the ball from $t = 0\,\text{s}$ to $t = 1\,\text{s}$? How about from $t = 0.5\,\text{s}$ to $t = 1\,\text{s}$? From $t = 0.9\,\text{s}$ to $t = 1\,\text{s}$? From $t = 0.99\,\text{s}$ to $t = 1\,\text{s}$? Conjecture about the instantaneous vertical velocity of the ball at $t = 1\,\text{s}$.

$$\begin{aligned}
\text{Average Velocity on } [0, 1] &= \frac{s(0) - s(1)}{1 - 0} = 16\,\text{ft/s} \\
\text{Average Velocity on } [0.5, 1] &= \frac{s(0.5) - s(1)}{1 - 0} = 8\,\text{ft/s} \\
\text{Average Velocity on } [0.9, 1] &= \frac{s(0.9) - s(1)}{1 - 0} = 1.6\,\text{ft/s} \\
\text{Average Velocity on } [0.99, 1] &= \frac{s(0.99) - s(1)}{1 - 0} \approx 0.0016\,\text{ft/s}
\end{aligned}$$

We conjecture that the instantaneous vertical velocity of the ball at $t = 1\,\text{s}$ is $0\,\text{ft/s}$. Indeed, this makes sense as $t = 1$ corresponds to the vertex of the parabola $y = s(t)$, which is exactly when the ball stops moving upward and starts moving back downward.

Definition. Given a function f defined on an interval $[a, b]$, the **instantaneous rate of change at $x = a$** is

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Remark. The instantaneous rate of change tells us the slope of the tangent line (to the curve $y = f(x)$) at the point $(a, f(a))$.

Example 2.1.2. Find the equation of the line tangent to the curve $y = -x^2$ at the point $(3, -9)$.

$$\begin{aligned}
\lim_{x \to 3} \frac{-x^2 + 3^2}{x - 3} &= \lim_{x \to 3} \frac{-(x - 3)(x + 3)}{x - 3} \\
&= \lim_{x \to 3} -(x + 3) \\
&= -(3 + 3) = -6.
\end{aligned}$$

Using the point-slope form of a line with slope 6 passing through the point $(3, -9)$, we have

$$\begin{aligned} y + 9 &= 6(x - 3) \\ \Rightarrow y &= 6x - 27 \end{aligned}$$

is the equation of the tangent line we wanted.

Definition. The **derivative of a function** f **at a number** a, denoted $f'(a)$, is given by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

provided the limit exists.

By making the substitution $h = x - a$, we get the following equivalent definition:

Definition. The **derivative of a function** f **at a number** a, denoted $f'(a)$, is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

provided the limit exists.

Remark. Here the h represents the distance away from the point a. For reasons that we will see in the next chapter, this latter definition is the more common of the two definitions of a derivative at a point.

Example 2.1.3. Let $g(x) = \dfrac{2}{x} - 4$. Find $g'(1)$ and $g'(-3)$.

We'll use the first definition to find $g'(1)$ and the second definition to find $g'(-3)$.

$$\begin{aligned} g'(1) &= \lim_{x \to 1} \frac{\left(\frac{2}{x} - 4\right) - \left(\frac{2}{1} - 4\right)}{x - 1} \\ &= \lim_{x \to 1} \frac{\left(\frac{2}{x} - 2\right)}{x - 1} \\ &= \lim_{x \to 1} \frac{\left(\frac{2 - 2x}{x}\right)}{x - 1} \\ &= \lim_{x \to 1} \frac{-2(x - 1)}{x(x - 1)} \\ &= \lim_{x \to 1} \frac{-2}{x} \\ &= -2 \end{aligned}$$

and

$$\begin{aligned}
g'(-3) &= \lim_{h\to 0} \frac{\left(\frac{2}{-3+h} - 4\right) - \left(\frac{2}{-3} - 4\right)}{h} \\
&= \lim_{h\to 0} \frac{\left(\frac{2}{-3+h} + \frac{2}{3}\right)}{h} \\
&= \lim_{h\to 0} \frac{\left(\frac{2(3)+2(-3+h)}{3(-3+h)}\right)}{h} \\
&= \lim_{h\to 0} \frac{2h}{h(-9+h)} \\
&= \lim_{h\to 0} \frac{2}{-9+h} \\
&= -\frac{2}{9}.
\end{aligned}$$

Example 2.1.4. Find $f'(a)$ for the function $f(x) = \sqrt{1-3x}$. [Here we are assuming that $a < \frac{1}{3}$.]

Using the definition of the derivative at a point a, we have

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{\sqrt{1-3(a+h)} - \sqrt{1-3a}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{1-3(a+h)} - \sqrt{1-3a}}{h} \left(\frac{\sqrt{1-3(a+h)} + \sqrt{1-3a}}{\sqrt{1-3(a+h)} + \sqrt{1-3a}}\right) \\
&= \lim_{h\to 0} \frac{[1-3(a+h)] - [1-3a]}{h\left(\sqrt{1-3(a+h)} + \sqrt{1-3a}\right)} \\
&= \lim_{h\to 0} \frac{-3h}{h\left(\sqrt{1-3(a+h)} + \sqrt{1-3a}\right)} \\
&= \lim_{h\to 0} \frac{-3}{\sqrt{1-3(a+h)} + \sqrt{1-3a}} \\
&= \frac{-3}{2\sqrt{1-3a}}.
\end{aligned}$$

2.2 The Derivative as a Function

Definition. Given a function f, the **derivative of** f is defined to be the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

The derivative is a function that tells you the slope of the tangent line at every point along the curve. In a physical system, the first derivative of the position function is the *velocity function*.

Remark. There are several equivalent notations for the derivative of $y = f(x)$. They are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x),$$

and we will be using them (notably the first four) interchangeably.

Example 2.2.1. Let $f(x) = \sqrt{x+2}$. Find the derivative $\frac{df}{dx}$.

$$\begin{aligned}
\frac{df}{dx} &= \lim_{h \to 0} \frac{\sqrt{(x+h)+2} - \sqrt{x+2}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{(x+h)+2} - \sqrt{x+2}}{h} \left(\frac{\sqrt{(x+h)+2} + \sqrt{x+2}}{\sqrt{(x+h)+2} + \sqrt{x+2}} \right) \\
&= \lim_{h \to 0} \frac{(x+h+2) - (x+2)}{h(\sqrt{(x+h)+2} + \sqrt{x+2})} \\
&= \lim_{h \to 0} \frac{h}{h(\sqrt{(x+h)+2} + \sqrt{x+2})} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{(x+h)+2} + \sqrt{x+2}} \\
&= \frac{1}{2\sqrt{x+2}}.
\end{aligned}$$

Definition. A function f is **differentiable at** a if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) if it is differentiable at every number in the interval.

Example 2.2.2. Where is $f(x) = |x|$ differentiable? *[Hint: consider separately the cases when $x < 0$, $x = 0$, $x > 0$, and also the limits as $h \to 0^+$ and $h \to 0^-$.]*

It's up to the reader to prove that $f'(x)$ exists when $x \neq 0$. The interesting case happens when $x = 0$. If $f'(0)$ exists, the limit (in the definition of the derivative) must exist. Examining the left and right limits, we see that

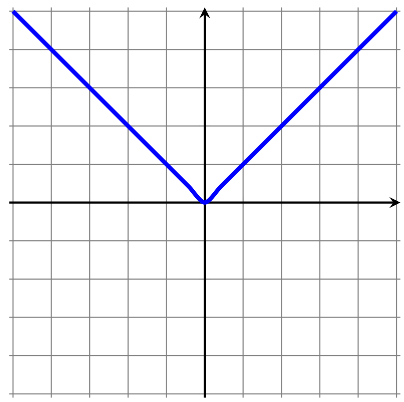
$$\lim_{x \to 0^-} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0^-} \frac{-x}{x} = -1$$

and

$$\lim_{x \to 0^+} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0^+} \frac{x}{x} = 1,$$

so since the limits do not agree, $f'(0)$ does not exist and thus f is differentiable on $(-\infty, 0) \cup (0, \infty)$.

Below is a graph of $f(x) = |x|$. Notice what the graph of the function looks like at the single point of non-differentiability.



This previous example tells us that functions with *cusps* are not differentiable at these cusps. The following result tells us another way in which a function can fail to be differentiable at a point.

Theorem 2.2.3. *If f is differentiable at a, then f is continuous at a.*

Proof. The proof of this result is an ε-δ argument that we won't go into. Heuristically, it comes down to the fact that the numerator (in the definition of a derivative at a point) implies that $\lim_{x \to a} f(x) = f(a)$; without this, the limit (in the definition of a derivative at a point) simply would not exist. $\square$

Corollary 2.2.4. *If f is discontinuous at a point, f is not differentiable at that point.*

Example 2.2.5. Where is $f(x) = \begin{cases} 2x + 7 & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}$ differentiable?

As we saw in class, if we naïvely took the left and right limits, we might be inclined to say that f is differentiable at 1 as

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = 2.$$

However, $f(x)$ is clearly discontinuous at $x = 1$, Corollary 2.2.4 tells us that f is not differentiable at $x = 1$. For the case when $x < 1$, we have

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{[2(x + h) + 7] - [2x + 7]}{h} = 2,$$

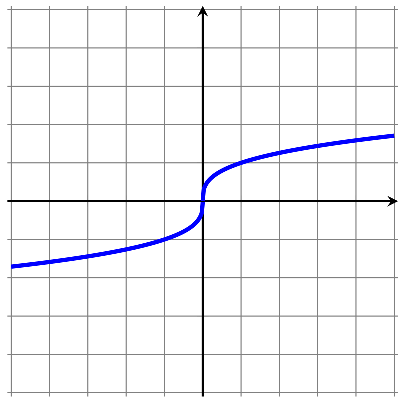
so f is differentiable on $(-\infty, 1)$. Also, when $x > 1$, we have

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} = 2x,$$

so f is differentiable on $(1, \infty)$ as well. Therefore f is differentiable on $(-\infty, 1) \cup (1, \infty)$.

There is another way that we can tell graphically if a function is differentiable at a point. If the tangent line is vertical, this corresponds to a derivative that would be ∞ or $-\infty$ (which means the derivative does not exist).

Example 2.2.6. Graph the function $f(x) = \sqrt[3]{x}$. Where, if anywhere, does f fail to be differentiable? Using the definition of the derivative, check your answer.



It looks like there might be a problem at $x = 0$. Indeed

$$\begin{aligned} f'(0) &= \lim_{x \to 0} \frac{\sqrt[3]{x} - \sqrt[3]{0}}{x - 0} \\ &= \lim_{x \to 0} \frac{\sqrt[3]{x}}{x} \\ &= \lim_{x \to 0} \frac{1}{x^{2/3}} = \text{Does Not Exist.} \end{aligned}$$

So $f(x) = \sqrt[3]{x}$ is differentiable on $(-\infty, 0) \cup (0, \infty)$.

Definition. The **second derivative of** f is the function $f'' = (f')'$. It is the derivative of the derivative f'. In Leibniz notation, we write $\dfrac{d^2 f}{dx^2}$ or $\dfrac{d^2 y}{dx^2}$.

The **third derivative of** f is the function $f''' = (f'')'$. It is the derivative of the second derivative f''. In Leibniz notation, we write $\dfrac{d^3 f}{dx^3}$ or $\dfrac{d^3 y}{dx^3}$.

The n^{th} **derivative of** f is the function $f^{(n)} = (f^{(n-1)})'$. It is the derivative of the $(n-1)^{\text{st}}$ derivative $f^{(n-1)}$. In Leibniz notation, we write $\dfrac{d^n f}{dx^n}$ or $\dfrac{d^n y}{dx^n}$.

Remark. In a physical system, the first derivative of the position function is the *velocity* function. The second derivative of the position function is the *acceleration* function. The third derivative is the *jerk* function. The fourth derivative is the *snap* function. The fifth derivative is the *crackle* function. The sixth derivative is the *pop* function.

Example 2.2.7. Find the fourth derivative $f^{(4)}(t)$ of the function $f(t) = t^4$.

First we find $f'(t)$:

$$\begin{aligned} f'(t) = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} &= \lim_{h\to 0} \frac{(t+h)^4 - t^4}{h} \\ &= \lim_{h\to 0} \frac{4t^3 + 6t^2h^2 + 4th^3 + h^4}{h} \\ &= \lim_{h\to 0} 4t^3 + 6t^2h + 4th^2t + h^3 \\ &= 4t^3. \end{aligned}$$

Now we find $f''(t)$:

$$\begin{aligned} f''(t) = \lim_{h\to 0} \frac{f'(t+h)-f'(t)}{h} &= \lim_{h\to 0} \frac{4(t+h)^3 + 4t^3}{h} \\ &= \lim_{h\to 0} \frac{12t^2h + 12th^2}{h} \\ &= \lim_{h\to 0} 12t^2 + 12th \\ &= 12t^2. \end{aligned}$$

Then we find $f'''(t)$:

$$\begin{aligned} f'''(t) = \lim_{h\to 0} \frac{f''(t+h)-f''(t)}{h} &= \lim_{h\to 0} \frac{12(t+h)^2 - 12t^2}{h} \\ &= \lim_{h\to 0} \frac{12th}{h} \\ &= \lim_{h\to 0} 12t \\ &= 12t. \end{aligned}$$

Finally we find $f^{(4)}(t)$:

$$\begin{aligned} f^{(4)}(t) = \lim_{h\to 0} \frac{f'''(t+h)-f'''(t)}{h} &= \lim_{h\to 0} \frac{12(t+h) - 12t}{h} \\ &= \lim_{h\to 0} \frac{12h}{h} \\ &= \lim_{h\to 0} 12 \\ &= 12. \end{aligned}$$

2.3 Basic Differentiation Formulas

As we saw previously, finding derivatives by taking limits is an absolute nightmare. Thankfully, there are some general patterns that arise that will make finding derivatives much faster for us.

Theorem 2.3.1 (Derivative of a Constant). *Let c be any real number. Then*

$$\frac{d}{dx}[c] = 0.$$

Proof. The proof is left as a very simple exercise. Just use the limit definition of a derivative. □

Theorem 2.3.2 (Power Rule for Derivatives). *Let n be any real number. Then*

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

We can see from Example 2.2.7 that this seems to be true (and indeed it is, I promise). When n is a nonnegative integer, the proof is again straightforward (although clunky given that you have to expand the binomial $(x+h)^n$).

Theorem 2.3.3 (Constant Multiple Rule for Derivatives). *Let c be any real number and $f(x)$ any function. Then*

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)].$$

Proof. The proof of this result follows from the constant multiple rule of limits. □

Theorem 2.3.4 (Sum/Difference Rule for Derivatives). *If $f(x)$ and $g(x)$ are both differentiable, then*

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)].$$

Proof. The proof of this result follows from the sum/difference rules for limits. □

Remark. Note that the product and quotient rules for derivatives do not behave as nicely as you might expect. We'll address these in a future lecture.

With these new rules, it now becomes very quick to find limits of things like polynomials and rational functions.

Example 2.3.5. Find $\dfrac{df}{dx}$ where $f(x) = x^{31} - 27x^2 + 18x + 1$.

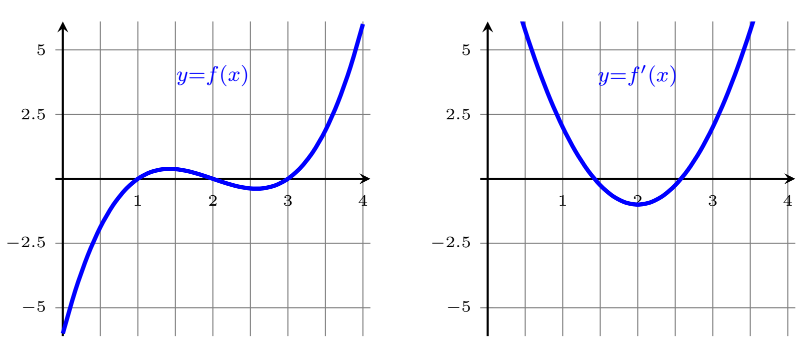
$$\begin{aligned}
\frac{df}{dx} &= \frac{d}{dx}\left[x^{31} - 27x^2 + 18x + 1\right] \\
&= \frac{d}{dx}\left[x^{31}\right] - \frac{d}{dx}\left[27x^2\right] + \frac{d}{dx}\left[18x\right] + \frac{d}{dx}[1] && \text{(sum/difference rule)} \\
&= \frac{d}{dx}\left[x^{31}\right] - 27\frac{d}{dx}\left[x^2\right] + 18\frac{d}{dx}\left[x\right] + \frac{d}{dx}[1] && \text{(constant multiple rule)} \\
&= 31x^{30} - 54x + 18 && \text{(power rule)}
\end{aligned}$$

Example 2.3.6. Find $\dfrac{dg}{dt}$ where $g(t) = \dfrac{t^{14} + 2t^7 + 13t^3 + 1}{t^5}$.

$$\begin{aligned}
\frac{dg}{dt} &= \frac{d}{dt}\left[\frac{t^{14} + 2t^7 + 13t^3 + 1}{t^5}\right] \\
&= \frac{d}{dt}\left[t^9 + 2t^2 + 13t^{-2} + t^{-5}\right] \\
&= \frac{d}{dt}\left[t^9\right] + \frac{d}{dt}\left[2t^2\right] + \frac{d}{dt}\left[13t^{-2}\right] + \frac{d}{dt}\left[t^{-5}\right] \\
&= 9t^8 + 4t - 26t^{-3} - 5t^{-6}
\end{aligned}$$

Example 2.3.7. Given $f(x) = x^3 - 6x^2 + 11x - 6$, sketch a graph f and f'.

First we find f'. By the power rule and sum/difference rules, we have that $f'(x) = 3x^2 - 12x + 11$. Notice the relationship between points with horizontal tangents in $f(x)$ correspond to x-intercepts of $f'(x)$. We also have a correspondence between positive (*resp.* negative) slopes of tangent lines of $f(x)$ with positive (*resp.* negative) y-values of $f'(x)$. This means that, given a graph of a function and its derivative, we should be able to determine which is which.



Proposition 2.3.8 (Derivative of Sine/Cosine)**.**

$$\frac{d}{dx}[\sin(x)] = \cos(x) \quad \textit{and} \quad \frac{d}{dx}[\cos(x)] = -\sin(x).$$

Proof. Recall from Section 1.4 that

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1 \quad \text{and} \quad \lim_{h\to 0} \frac{\cos(h)-1}{h} = 0.$$

$$\begin{aligned}
\frac{d}{dx}[\sin(x)] &= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0} \frac{[\sin(x)\cos(h)+\sin(h)\cos(x)]-\sin(x)}{h} && \text{(angle sum/difference identity)} \\
&= \lim_{h\to 0} \frac{\sin(x)[\cos(h)-1]+\sin(h)\cos(x)}{h} \\
&= \lim_{h\to 0} \sin(x)\frac{\cos(h)-1}{h} + \lim_{h\to 0} \frac{\sin(h)}{h}\cos(x) \\
&= \sin(x)(0) + (1)\cos(x) \\
&= \cos(x).
\end{aligned}$$

And similarly,

$$\begin{aligned}
\frac{d}{dx}[\cos(x)] &= \lim_{h\to 0} \frac{\cos(x+h)-\cos(x)}{h} \\
&= \lim_{h\to 0} \frac{[\cos(x)\cos(h)-\sin(x)\sin(h)]-\cos(x)}{h} && \text{(angle sum/difference identity)} \\
&= \lim_{h\to 0} \frac{\cos(x)[\cos(h)-1]-\sin(h)\sin(x)}{h} \\
&= \lim_{h\to 0} \cos(x)\frac{\cos(h)-1}{h} - \lim_{h\to 0} \frac{\sin(h)}{h}\sin(x) \\
&= \cos(x)(0) - (1)\sin(x) \\
&= -\sin(x).
\end{aligned}$$

□

2.4 The Product and Quotient Rules

From last time we had some nice properties of derivatives - we could differentiate across scalar multiplication and addition (formally, we say that $\frac{d}{dx}$ is a "linear operator"). However, as we will see, derivatives of products and quotients do not behave quite as obviously as derivatives of sums and differences.

Theorem 2.4.1 (Product Rule)**.** *Let f and g be differentiable functions. Then*

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

Proof. Again, using the limit definition of the derivative, we have

$$\begin{aligned}
&\frac{d}{dx}[f(x)g(x)] \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h) - f(x)]g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)[g(x+h) - g(x)]}{h} \\
&= \left[\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right] \cdot \left[\lim_{h\to 0} g(x+h)\right] + \left[\lim_{h\to 0} f(x)\right] \cdot \left[\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right] \\
&= f'(x)g(x) + f(x)g'(x).
\end{aligned}$$

□

Example 2.4.2. Let $h(x) = (2x^3 + 7)(x - 5\sqrt{x})$. Find $h'(x)$ using the product rule. Check your answer by first expanding out the function (FOIL) and then taking the derivative.

We first identify two functions $f(x) = 2x^3 + 7$ and $g(x) = x - 5\sqrt{x} = x - 5x^{1/2}$ so that $h(x) = f(x)g(x)$. Now,

$$f'(x) = \frac{d}{dx}[2x^3 + 7] = 2\frac{d}{dx}[x^3] + \frac{d}{dx}[7] = 2[3x^2] + 0 = 6x^2,$$

and

$$g'(x) = \frac{d}{dx}[x - 5x^{1/2}] = \frac{d}{dx}[x] - 5\frac{d}{dx}[x^{1/2}] = 1 - 5\left[\frac{1}{2}x^{-1/2}\right] = 1 - \frac{5}{2}x^{-1/2}.$$

Thus, by the product rule,

$$h'(x) = 6x^2\left(x - 5x^{1/2}\right) + \left(2x^3 + 7\right)\left(1 - \frac{5}{2}x^{-1/2}\right).$$

Unsurprisingly, the quotients of differentiable functions do not behave as obviously as we might like them to either.

Theorem 2.4.3 (Quotient Rule)**.** *Let f and g be differentiable functions. Then for any x where $g(x) \neq 0$,*

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

Proof. The proof of this theorem uses a similar trick as in the proof of the product rule: add $f(x)g(x) - f(x)g(x)$ in the numerator, then split up the limits. It is left as an exercise to the reader. □

Example 2.4.4. Let $s(t) = \dfrac{5}{t^3 - 9}$. Find $s'(2)$.

Let $f(t) = 5$ and $g(t) = t^3 - 9$ so that $s(t) = \dfrac{f(t)}{g(t)}$. Then

$$f'(t) = 0 \quad \text{and} \quad g'(t) = 3t^2.$$

So by the quotient rule,

$$s'(t) = \frac{0(t^3 - 9) - 5(3t^2)}{(t^2 - 9)^2} = \frac{-15t^2}{(t^2 - 9)^2}$$

and thus

$$s'(2) = -\frac{15(2)^2}{[(2)^2 - 9]^2} = -60.$$

Example 2.4.5. Let f and g be differentiable functions, and let $F(x) = f(x)g(x)$ and $G(x) = \dfrac{f(x)}{g(x)}$. Suppose $f(-1) = 5$, $f'(-1) = 12$, $g(-1) = -3$, and $g'(-1) = 8$. Find $F'(-1)$ and $G'(-1)$.

The product rule tells us that

$$\begin{aligned} F'(-1) &= f'(-1)g(-1) + f(-1)g'(-1) \\ &= (12)(-3) + (5)(8) \\ &= -36 + 40 \\ &= 4. \end{aligned}$$

The quotient rule tells us that

$$\begin{aligned} G'(-1) &= \frac{f'(-1)g(-1) - f(-1)g'(-1)}{[g(-1)]^2} \\ &= \frac{(12)(-3) - (5)(8)}{[-3]^2} \\ &= -\frac{76}{9}. \end{aligned}$$

Since $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$, $\csc(x) = \dfrac{1}{\sin(x)}$, $\sec(x) = \dfrac{1}{\cos(x)}$, and $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$, we can now use the quotient rule to complete our list of derivatives of trigonometric functions.

Proposition 2.4.6 (Derivatives of Trigonometric Functions)**.**

$$\frac{d}{dx}[\sin(x)] = \cos(x) \qquad \frac{d}{dx}[\cos(x)] = -\sin(x)$$
$$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x) \qquad \frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$$
$$\frac{d}{dx}[\tan(x)] = \sec^2(x) \qquad \frac{d}{dx}[\cot(x)] = -\csc^2(x)$$

Proof. We'll obtain the derivative of $\tan(x)$, and leave the remaining derivatives as an exercise for the reader.

Let $f(x) = \sin(x)$ and $g(x) = \cos(x)$ so that $\tan(x) = \dfrac{\sin(x)}{\cos(x)} = \dfrac{f(x)}{g(x)}$. Then

$$f'(x) = \cos(x) \quad \text{and} \quad g'(x) = -\sin(x).$$

Thus, by the quotient rule

$$\begin{aligned}
\frac{d}{dx}[\tan(x)] = \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] &= \frac{\cos(x)\cos(x) - \sin(x)[-\sin(x)]}{\cos^2(x)} \\
&= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} \qquad (\text{since } \cos^2\theta + \sin^2\theta = 1) \\
&= \sec^2(x).
\end{aligned}$$

□

Example 2.4.7. Find $\frac{dh}{dx}$ where $h(x) = 7x^2\,[2\tan(x) + 3\sec(x)]$.

Again, let $f(x) = 7x^2$ and $g(x) = 2\tan(x) + 3\sec(x)$. Then

$$f'(x) = 14x \quad \text{and} \quad g'(x) = 2\sec^2(x) + 3\sec(x)\tan(x).$$

So, by the product rule, we have that

$$h'(x) = f'(x)g(x) + f(x)g'(x) = 14x[2\tan(x) + 3\sec(x)] + 7x^2\left[2\sec^2(x) + 3\sec(x)\tan(x)\right].$$

Example 2.4.8. Find $f'(x)$ for $f(x) = \dfrac{x^3 \sin(x)}{x+1}$.

Notice that our numerator is a product of functions, so we're going to have to apply a product rule within the quotient rule.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left[\frac{x^3 \sin(x)}{x+1}\right] \\
&= \frac{\frac{d}{dx}[x^3 \sin(x)](x+1) - x^3 \sin(x)\frac{d}{dx}[x+1]}{(x+1)^2} && \text{(quotient rule)} \\
&= \frac{\left(\frac{d}{dx}[x^3]\sin(x) + x^3 \frac{d}{dx}[\sin(x)]\right)(x+1) - x^3 \sin(x)\frac{d}{dx}[x+1]}{(x+1)^2} && \text{(product rule)} \\
&= \frac{3x^2 \sin(x)(x+1) + x^3 \cos(x)(x+1) - x^3}{(x+1)^2}.
\end{aligned}$$

Example 2.4.9. Find $f'(\theta)$ for $f(\theta) = \sin^2\theta$.

$$\begin{aligned}
f'(\theta) &= \frac{d}{d\theta}[\sin\theta \sin\theta] \\
&= \frac{d}{d\theta}[\sin\theta]\sin\theta + \sin\theta\frac{d}{d\theta}[\sin\theta] && \text{(product rule)} \\
&= \cos\theta\sin\theta + \sin\theta\cos\theta \\
&= 2\sin\theta\cos\theta.
\end{aligned}$$

Example 2.4.10. Find $f'(\theta)$ for $f(\theta) = \sin^3\theta$.

$$\begin{aligned}
f'(\theta) &= \frac{d}{d\theta}\left[\sin^3\theta\right] \\
&= \frac{d}{d\theta}\left[\sin\theta\sin^2\theta\right] \\
&= \frac{d}{d\theta}[\sin\theta]\sin^2\theta + \sin\theta\frac{d}{d\theta}\left[\sin^2\theta\right] \\
&= \frac{d}{d\theta}[\sin\theta]\sin^2\theta + \sin\theta\frac{d}{d\theta}[\sin\theta\sin\theta] \\
&= \frac{d}{d\theta}[\sin\theta]\sin^2\theta + \sin\theta\left(\frac{d}{d\theta}[\sin\theta]\sin\theta + \sin\theta\frac{d}{d\theta}[\sin\theta]\right) \\
&= \cos\theta\sin^2\theta + \sin\theta\left(\cos\theta\sin\theta + \sin\theta\cos\theta\right) \\
&= 3\sin^2\theta\cos\theta.
\end{aligned}$$

There appears to be a pattern forming here. We conjecture that $\dfrac{d}{d\theta}[\sin^n\theta] = n\sin^{n-1}\theta\cos\theta$. Indeed, if we think about $\sin^n(x) = f(g(x))$ where $f(x) = x^n$ and $g(x) = \sin(x)$, we see that somehow it's like there's a combination of the derivatives $f'(x) = nx^{n-1}$ and $g'(x) = \cos(x)$ involved in the derivative of $f(g(x))$. We'll make this formal in the next section.

2.5 The Chain Rule

What if you were asked to find the derivative of a composite function $f(g(x))$? Certainly you could approach with limits, but limits are extremely messy and it'd be much nicer if we had a rule that gave us an all-inclusive approach to composite functions. Indeed, there is such a rule:

Theorem 2.5.1 (Chain Rule)**.** *Suppose f and g are both differentiable functions. Then the composite $(f \circ g)(x) = f(g(x))$ is differentiable and the derivative is given by*

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, letting $y = f(u)$ and $u = g(x)$, we have that $y = f(g(x))$ and so we would write

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Remark. When it comes to the chain rule, often times the most difficult part is determining what the two functions that form your composite function are.

Example 2.5.2. Find the derivative of $h(x) = \tan(3x)$

Let $f(x) = \tan(x)$ and $g(x) = 3x$ so that $h(x) = f(g(x))$. Then

$$f'(x) = \sec^2(x)$$

and

$$g'(x) = 3,$$

so

$$h'(x) = f'(g(x)) \cdot g'(x) = \sec^2(g(x)) \cdot 3 = 3\sec^2(3x).$$

Example 2.5.3. Find $\dfrac{dh}{dt}$ where $h(t) = (t^2 - 7)^{861547}$.

Let $f(t) = t^{861}$ and $g(t) = (t^2 - 7)$ so that $h(t) = f(g(t))$. Then

$$f'(t) = 861t^{860}$$

and

$$g'(t) = 2t,$$

so

$$h'(t) = f'(g(t)) \cdot g'(t) = 861(t^2 - 7)^{860} \cdot 2t.$$

Sometimes you may need to use the chain rule in conjunction with the product or quotient rules.

Example 2.5.4. Find $r'(\theta)$ where $r(\theta) = \sqrt{(\theta+1)\sin\theta}$.

Let $f(\theta) = \sqrt{\theta}$, and $g(\theta) = (\theta+1)\sin\theta$. Then

$$f'(\theta) = \frac{1}{2}\theta^{-1/2}$$

and, using the product rule, we have

$$g'(\theta) = \sin\theta + (\theta+1)\cos\theta,$$

so using the chain rule

$$r'(\theta) = f'(g(\theta)) \cdot g'(\theta) = \frac{1}{2}\left[(\theta+1)\sin\theta\right]^{-1/2}\left[\sin\theta + (\theta+1)\cos\theta\right].$$

Example 2.5.5. Find the derivative of $F(x) = \dfrac{\cos(\sqrt{x})}{\csc(x)}$.

Let $f(x) = \cos(x)$, $g(x) = \sqrt{x}$ and $h(x) = \csc(x)$. Then the numerator is $f(g(x))$ and the denominator is $h(x)$. We have that

$$f'(x) = -\sin(x)$$

and

$$g'(x) = \frac{1}{2}x^{-1/2},$$

so

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x) = -\frac{1}{2}\cos(\sqrt{x})x^{-1/2}.$$

We also have that

$$h'(x) = -\csc(x)\cot(x),$$

so using the quotient rule,

$$\begin{aligned} F'(x) &= \frac{(f \circ g)'(x)h(x) - (f \circ g)(x)h'(x)}{[h(x)]^2} \\ &= \frac{-\frac{1}{2}\cos(\sqrt{x})x^{-1/2}\csc(x) + \cos(\sqrt{x})\csc(x)\cot(x)}{\csc^2(x)}. \end{aligned}$$

Sometimes, we may even have to use an embedded chain rule *(chain rule-ception)*.

Example 2.5.6. Find the derivative $\dfrac{dT}{d\varphi}$ of $T(\varphi) = \sin(\tan(\csc\varphi))$. Let $f(\varphi) = \sin\varphi$, $g(\varphi) = \tan\varphi$, and $h(\varphi) = \csc\varphi$, so then $T(\varphi) = f(g(h(\varphi))$. We have that

$$f'(\varphi) = \cos\varphi,$$
$$g'(\varphi) = \sec^2\varphi,$$

and

$$h'(\varphi) = -\csc\varphi\cot\varphi.$$

So then

$$\begin{aligned} T'(\varphi) &= f'(g(h(\varphi))) \cdot (g \circ h)'(\varphi) \\ &= f'(g(h(\varphi)) \cdot g'(h(\varphi)) \cdot h'(\varphi) \\ &= \cos(\tan(\csc\varphi)) \cdot \sec^2(\csc\varphi) \cdot [-\csc\varphi\cot\varphi] \\ &= -\cos(\tan(\csc\varphi))\sec^2(\csc\varphi)\csc\varphi\cot\varphi \end{aligned}$$

Example 2.5.7. Find all points on the graph of $y = 1+\sqrt{8x^2 - x^4}$ where the tangent line is horizontal. Confirm your results by sketching a graph.

The slope of a horizontal tangent line is 0, so we're looking for places where $y' = 0$. Let $f(x) = 1+\sqrt{x}$ and $g(x) = 8x^2 - x^4$ so that $y = f(g(x))$. Then

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

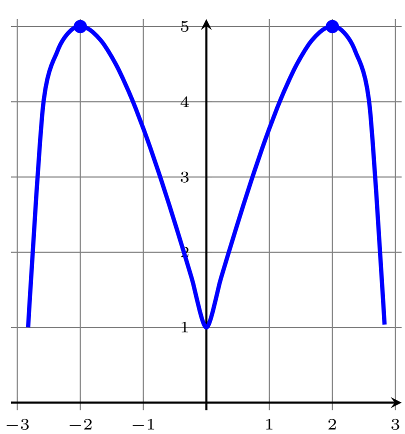
and

$$g'(x) = 16x - 4x^3,$$

so

$$y' = f'(g(x)) \cdot g'(x) = \frac{16x - 4x^3}{2\sqrt{8x^2 - x^4}} = \frac{4x(4 - x^2)}{2\sqrt{4x^2 - x^4}}.$$

We see that $y' = 0$ when $x = 0, \pm 2$. However, 0 is not in the domain of y', so in fact the only horizontal tangent lines occur when $x = \pm 2$.



2.6 Implicit Differentiation

Every function we've encountered up to this point can be described as one variable *explicilty* in terms of another variable, for example

$$y = \sqrt{x-1}$$
$$y = 47x^2 \sin(x)$$

However, there are some functions that may be defined *implicitly* by a relation between x and y, for example

$$x^2 + y^2 = 169 \qquad \text{(circle of radius 13)}$$
$$x = \frac{2}{3}y^2 \qquad \text{(parabola opening to the right)}$$
$$4(x^2 + y^2) = (x^2 + y^2 - 2xy)^2 \qquad \text{(cardioid)}$$

In some of these cases, it's easy to represent the implicit function as at least one explicit function, but in general that need not happen (as is the case with the cardioid, which requires a minimum of four explicit functions). In these cases, we would still like to be able to find the derivative $\frac{dy}{dx}$, say to find the equation of the tangent line. The key is to use the chain rule and treat $y = y(x)$ as a function of x.

Example 2.6.1. If $x^2 + y^2 = 400$, find $\dfrac{dy}{dx}$.

Since we're treating y as a function of x, we actually have that $y^2 = [y(x)]^2 = g(y(x))$, where $g = x^2$. Since this is a composite function, we'll need to use the chain rule. Indeed, we have that

$$\frac{d}{dx}[y^2] = \frac{d}{dx}[g(y(x))] = g'(y(x)) \cdot y'(x) = g'(y(x))\frac{dy}{dx} = 2[y(x)]\frac{dy}{dx} = 2y\frac{dy}{dx}.$$

Since equal functions have equal derivatives everywhere, we can take a derivative of both sides of our given relation

$$x^2 + y^2 = 400$$
$$\frac{d}{dx}\left[x^2 + y^2\right] = \frac{d}{dx}[400]$$
$$\frac{d}{dx}\left[x^2\right] + \frac{d}{dx}\left[y^2\right] = 0$$
$$2x + 2y\frac{dy}{dx} = 0$$

and now we can solve for $\frac{dy}{dx}$ as we would for any other variable

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}.$$

Example 2.6.2. Find $y'(x)$ implicitly for the equation $y = \sin(xy)$.

Taking a derivative of both sides of the given equation (with respect to x), we get

$$\begin{aligned}
\frac{d}{dx}[y] &= \frac{d}{dx}[\sin(xy)] \\
y'(x) &= \cos(xy)\frac{d}{dx}[xy] && \text{(chain rule)} \\
y'(x) &= \cos(xy)\,(y + xy'(x)) && \text{(product rule)} \\
\frac{dy}{dx} &= y\cos(xy) + x\cos(xy)y'(x) \\
-y\cos(xy) &= x\cos(xy)y'(x) - \frac{dy}{dx} \\
-y\cos(xy) &= (x\cos(xy) - 1)y'(x) \\
\Rightarrow \quad y'(x) &= \frac{-y\cos(xy)}{x\cos(xy) - 1}.
\end{aligned}$$

Example 2.6.3. Given the relation $x^2 - y^2 = 81$, find the second derivative $\frac{d^2y}{dx^2}$.

We begin by finding the first derivative

$$\begin{aligned}
\frac{d}{dx}\left[x^2 - y^2\right] &= \frac{d}{dx}[81] \\
2x - 2y\frac{dy}{dx} &= 0 \\
\Rightarrow \quad \frac{dy}{dx} &= \frac{x}{y}. && (2.6.1)
\end{aligned}$$

Now we take the derivative of each side of this new equation (with respect to x)

$$\begin{aligned}
\frac{d}{dx}\left[\frac{dy}{dx}\right] &= \frac{d}{dx}\left[\frac{x}{y}\right] \\
\frac{d^2y}{dx^2} &= \frac{y - x\frac{dy}{dx}}{y^2}. && (2.6.2)
\end{aligned}$$

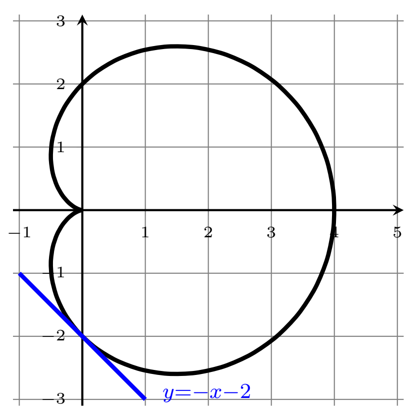
Now, we're not quite done yet as we want to represent the second derivative entirely in terms of x and y. So, we substitute Equation 2.6.1 into Equation 2.6.2 and get

$$\frac{d^2y}{dx^2} = \frac{y - x\frac{dy}{dx}}{y^2} = \frac{y - x\left(\frac{x}{y}\right)}{y^2} = \frac{y^2 - x}{y^3}.$$

Example 2.6.4. Find the equation of the tangent lines to the cardioid given by $4(x^2+y^2) = (x^2+y^2-2x)^2$ at the point $(0,-2)$ To find the horizontal tangent lines, we use implicit differentiation to find $\frac{dy}{dx}$ and then plug in $(0,-2)$ to find the slope of the tangent line. Taking a derivative of both sides of the given equation (with respect to x), we get

$$\begin{aligned}
\frac{d}{dx}\left[4(x^2+y^2)\right] &= \frac{d}{dx}\left[\left(x^2+y^2-2x\right)^2\right] \\
4\left(2x+2y\frac{dy}{dx}\right) &= 2\left(x^2+y^2-2x\right)\frac{d}{dx}\left[x^2+y^2-2x\right] \\
4\left(2x+2y\frac{dy}{dx}\right) &= 2\left(x^2+y^2-2x\right)\left(2x+2y\frac{dy}{dx}-2\right) \\
4\left(2x+2y\frac{dy}{dx}\right) &= 2\left(2x^3+2x^2y\frac{dy}{dx}-2x^2+2xy^2+2y^3\frac{dy}{dx}-2y^2-4x^2-4xy\frac{dy}{dx}+4x\right) \\
4\left(2x+2y\frac{dy}{dx}\right) &= 4\left(x^3+x^2y\frac{dy}{dx}-x^2+xy^2+y^3\frac{dy}{dx}-y^2-2x^2-2xy\frac{dy}{dx}+2x\right) \\
2x+2y\frac{dy}{dx} &= x^3+x^2y\frac{dy}{dx}-3x^2+xy^2+y^3\frac{dy}{dx}-y^2-2xy\frac{dy}{dx}+2x \\
2x-x^3+3x^2-xy^2+y^2-2x &= x^2y\frac{dy}{dx}+y^3\frac{dy}{dx}-2xy\frac{dy}{dx}-2y\frac{dy}{dx} \\
2x-x^3+3x^2-xy^2+y^2-2x &= \left(x^2y+y^3-2xy-2y\right)\frac{dy}{dx} \\
\Rightarrow \quad \frac{dy}{dx} &= \frac{2x-x^3+3x^2-xy^2+y^2-2x}{x^2y+y^3-2xy-2y}.
\end{aligned}$$

Plugging in $x=0$ and $y=-2$, we get that $\frac{dy}{dx}=-1$ and thus the equation of our tangent line through $(0,-2)$ is $y=-x-2$.



2.7 Related Rates

We will motivate this topic with the following example.

Example 2.7.1. Suppose water is being drained out of a conical tank. Given that the volume of a cone is $V = \frac{\pi}{3}r^2h$, rate of change in the volume of water, $\frac{dV}{dt}$, should be related to both the rate of change of the radius of the water's surface, $\frac{dr}{dt}$, and the rate of change of the height of the water, $\frac{dh}{dt}$. Indeed, since $V = V(t)$, $r = r(t)$, and $h = h(t)$ are all functions of time, we can use implicit differentiation to get

$$\begin{aligned}\frac{d}{dt}[V] &= \frac{d}{dt}\left[\frac{\pi}{3}r^2h\right] \\ \frac{dV}{dt} &= \frac{\pi}{3}\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right) && \text{(product rule).}\end{aligned}$$

So now we know how the rates $\frac{dV}{dt}$, $\frac{dr}{dt}$, and $\frac{dh}{dt}$ are related. We may call this equation a **related rates** equation.

Example 2.7.2. A stone is dropped into a calm lake, which causes concentric circular ripples to emanate from the splash point. The radius r of the outermost ripple is increasing a rate of 2 feet per second. When the radius gets to be 7 feet, at what rate is the total area A of the rippled water changing?

Recall that $A = \pi r^2$. Using implicit differentiation, we see that the changing area is related to the changing radius by

$$\begin{aligned}\frac{d}{dt}[A] &= \frac{d}{dt}\left[\pi r^2\right] \\ \frac{dA}{dt} &= 2\pi r\frac{dr}{dt}.\end{aligned}$$

We're given that $\frac{dr}{dt} = 2\,\text{ft/s}$, so when $r = 7\,\text{ft}$, we have

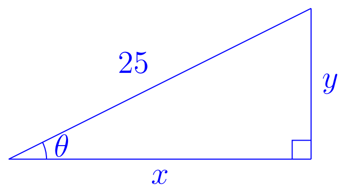
$$\frac{dA}{dt} = 2\pi(7)(2) = 28\pi\,\text{ft}^2/\text{s}.$$

Example 2.7.3. A 25-foot ladder is leaning against the wall of a building. The base of the ladder is being pulled away from the building at a rate of 2 feet per second, and the top of the ladder is sliding down the wall.

a. How fast is the top of the ladder sliding down the wall when the base is 8 feet away from the wall?

b. At what rate is the angle between the ladder and the ground changing when the base is 8 feet away from the wall?

Let $x = x(t)$ represent the distance of the base of the ladder from the wall, $y = y(t)$ be the height of the top of the latter, and $\theta = \theta(t)$ the angle formed between the ground and the base of the ladder.



a. We want to relate $\frac{dx}{dt}$ and $\frac{dy}{dt}$. From the picture above, it's clear that x and y are related by

$$x^2 + y^2 = 25^2 = 625.$$

With implicit differentiation, we have that

$$\begin{aligned} \frac{d}{dt}\left[x^2+y^2\right] &= \frac{d}{dt}[625] \\ 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 0 \\ \Rightarrow \qquad \frac{dy}{dt} &= -\frac{x}{y}\frac{dx}{dt} = -\frac{x}{\sqrt{625-x^2}}\frac{dx}{dt}. \end{aligned}$$

We're given that $\frac{dx}{dt} = 2\,\text{ft/s}$, so when $x = 8$, we have

$$\frac{dy}{dt} = -\frac{2(8)}{2\sqrt{625-64}}(2) = -\frac{16}{\sqrt{561}} \approx -0.676\,\text{ft/s}.$$

b. Now we want to relate $\frac{dx}{dt}$ and $\frac{d\theta}{dt}$. Again, from the picture above, we have that x, y, and θ are related by $\cos\theta = \frac{x}{25}$ and $\sin\theta = \frac{y}{25}$. With implicit differentiation,

$$\begin{aligned} \frac{d}{dt}[\cos\theta] &= \frac{d}{dt}\left[\frac{x}{25}\right] \\ -\sin\theta\frac{d\theta}{dt} &= \frac{1}{25}\frac{dx}{dt} \\ \Rightarrow \qquad \frac{d\theta}{dt} &= -\frac{1}{25\sin\theta}\frac{dx}{dt} = -\frac{1}{y}\frac{dx}{dt} = -\frac{1}{\sqrt{625-x^2}}\frac{dx}{dt}. \end{aligned}$$

We're given that $\frac{dx}{dt} = 2\,\text{ft/s}$, so when $x = 8$, we get

$$\frac{d\theta}{dt} = -\frac{1}{\sqrt{625-64}}(2) = -\frac{2}{\sqrt{561}} \approx -0.084\,\text{rad/s} \approx -4.838\,\text{deg/s}.$$

Example 2.7.4. A perfectly spherical balloon is being filled with air at a constant rate of 10 cubic inches per minute. At some point in time, an observer measures that the radius is increasing at a rate of 1.7 inches per minute. What is the radius of the balloon when this measurement is taken, and what is the volume of the balloon when this measurement is taken?

Let $r = r(t)$ be the radius of the balloon and $V = V(t)$ the volume of the balloon at time t. Recall that the volume of a sphere is given by

$$V = \frac{4}{3}\pi r^3.$$

So, differentiating both sides of this equation with respect to t, we have

$$\begin{aligned} \frac{d}{dt}[V] &= \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right] \\ \frac{dV}{dt} &= 4\pi r^2 \frac{dr}{dt}. \end{aligned} \tag{2.7.1}$$

We're given that $\frac{dV}{dt} = 10\,\text{in}^3/\text{min}$ and $\frac{dr}{dt} = 1.7\,\text{in/min}$, so rearranging Equation 2.7.1 to solve for r, we get that

$$\begin{aligned} r &= \sqrt{\frac{\frac{dV}{dt}}{4\pi\frac{dr}{dt}}} \\ &= \sqrt{\frac{10}{4\pi(1.7)}} \\ &\approx 0.684\,\text{in}, \end{aligned}$$

at the time the measurement is taken. The equation for the volume of the sphere tells us that the balloon's volume is

$$V \approx \frac{4}{3}\pi(0.684) \approx 1.34\,\text{in}^3$$

at the time of the measurement.

Example 2.7.5. A plane is flying due north at a constant 500 kilometers per hour and another is flying due east at a constant 600 kilometers per hour. At what rate is the distance between the planes changing after two hours have passed?

Let $y = y(t)$ be the distance traveled north and $x = x(t)$ be the distance traveled east after t hours have passed. By the Pythagorean Theorem, we have that the distance $z = z(t)$ between the two planes after t hours is given by

$$z^2 = x^2 + y^2.$$

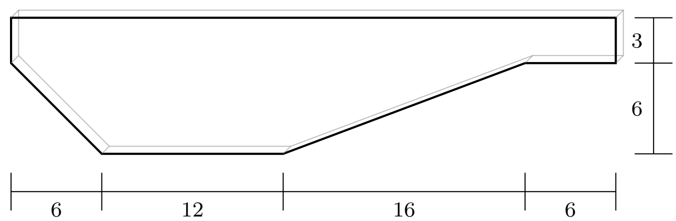
Differentiating both sides with respect to t, we have

$$\begin{aligned} \frac{d}{dt}\left[z^2\right] &= \frac{d}{dt}\left[x^2 + y^2\right] \\ 2z\frac{dz}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ \Rightarrow \quad \frac{dz}{dt} &= \frac{x}{z}\frac{dx}{dt} + \frac{y}{z}\frac{dy}{dt} \\ &= \frac{x}{\sqrt{x^2+y^2}}\frac{dx}{dt} + \frac{y}{\sqrt{x^2+y^2}}\frac{dy}{dt}. \qquad (2.7.2) \end{aligned}$$

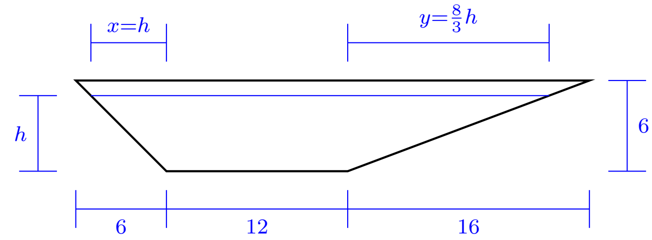
After 2 hours, we have that $x = 1200\,\text{km}$ and $y = 1000\,\text{km}$. Since we're given that $\frac{dx}{dt} = 600\,\text{km/h}$ and $\frac{dy}{dt} = 500\,\text{km/h}$, we plug all of this into Equation 2.7.2 to get

$$\frac{dz}{dt} = \frac{1200}{\sqrt{1200^2 + 1000^2}}(600) + \frac{1000}{\sqrt{1200^2 + 1000^2}}(500) \approx 781\,\text{km/h}.$$

Example 2.7.6. A swimming pool is 20 ft wide, 40 ft long, 4 ft deep at the shallow end, and 9 ft deep at its deepest point. A cross-section is shown in the figure below. If the pool is being filled at a rate of $0.8\,\text{ft}^3/\text{min}$, how fast is the water level rising when the depth at the deepest point is 5 ft?



Notice that the rate of change of the volume will be different when the water level is above 6 ft and when it is below 6 ft. As such, we are in the latter case. Let $h = h(t)$ be the height of the water.



(Here x and y were determined by similar triangles). We thus have that the volume $V = V(t)$ of the pool at height h is given by

$$V = 20\left[\frac{1}{2}(h)h + \frac{1}{2}\left(\frac{8}{3}h\right)h + 12h\right] = \frac{110}{3}h^2 + 240h.$$

Differentiating both sides with respect to t yields

$$\begin{aligned}
\frac{d}{dt}[V] &= \frac{d}{dt}\left[\frac{110}{3}h^2 + 240h\right] \\
\frac{dV}{dt} &= \frac{220}{3}h\frac{dh}{dt} + 240\frac{dh}{dt} \\
&= \left[\frac{220}{3}h + 240\right]\frac{dh}{dt} \\
\Rightarrow \quad \frac{dh}{dt} &= \frac{1}{\frac{220}{3}h + 240}\frac{dV}{dt}.
\end{aligned}$$

We're given that $\frac{dV}{dt} = 0.8\,\text{ft}^3/\text{min}$, so when the water level is 5 ft, we get

$$\frac{dh}{dt} = \frac{1}{\frac{220}{3}(5) + 240}(0.8) \approx 0.00132\,\text{ft}/\min.$$

2.8 Linear Approximations and Differentials

As we have seen, given a function f and a point a in the domain, the tangent line at the point $(a, f(a))$ is a fairly accurate approximation of the function values near a. At the point $(x_0, y_0) = (a, f(a))$, the equation of the tangent line is given by

$$\begin{aligned}(y - y_0) &= m(x - x_0)\\(y - f(a)) &= f'(a)(x - a)\\\Rightarrow \quad y &= f(a) + f'(a)(x - a).\end{aligned}$$

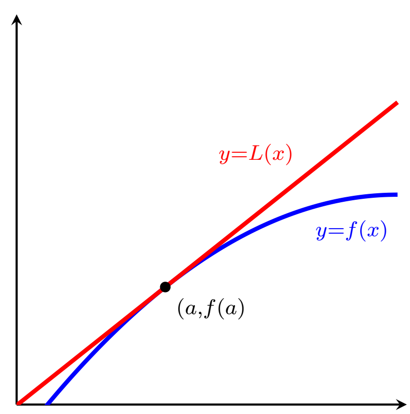
Definition. For all x near a point a, the approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the **linear approximation** or **tangent line approximation** of f at a. The linear function

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a.



Example 2.8.1. Use the tangent line approximation of

$$f(x) = 1 + \sin(x)$$

at the point $(0, 1)$ to approximate the value of $f(0.1)$. How does your approximation compare to the calculator's given value of $f(0.1)$?
We're doing the tangent line approximation at $(0, 1)$, so $a = 0$. Hence

$$f(x) \approx f(0) + f'(0)(x - 0).$$

A quick computation shows that $f'(x) = \cos(x)$, so $f'(0) = 1$. Hence

$$f(x) \approx 1 + 1(x - 0) = 1 + x.$$

hence

$$f(0.1) \approx 1 + 0.1 = 1.1.$$

Indeed, according to our calculator, $f(0.1) = 1.099833\ldots$, so our approximation is very close.

Example 2.8.2. Use the linearization of

$$f(x) = \sqrt{x}$$

at 16 to approximate $\sqrt{16.5}$. How does your approximation compare to the calculator's given value of $\sqrt{16.5}$?

We're looking at a linear approximation of f at 16, so $a = 16$. The linearization is given by

$$L(x) = f(a) + f'(a)(x - a).$$

A quick computation shows that $f'(x) = \frac{1}{2}x^{-1/2}$, so $f'(16) = \frac{1}{8}$, hence

$$L(x) = 4 + \frac{1}{8}(x - 16).$$

So, $f(16.5) \approx L(16.5)$, whence we compute

$$L(16.5) = 4 + \frac{1}{8}(16.5 - 16) = 4.0625.$$

Indeed, according to our calculator, $f(16.5) = 4.0620192\ldots$, so our approximation is very close.

Why do we care about linear approximations? Functions can be very computationally complex, but linear functions are incredibly simple. So, we can approximate values of a rather complicated function with simpler linear functions to any degree of accuracy we need.

Approximations can also be thought of in terms of *differentials*.

Definition. If $y = f(x)$, where f is a differentiable function, then the **differential**, dx, is an independent variable and can take on any real number, and the **differential**, dy is defined by

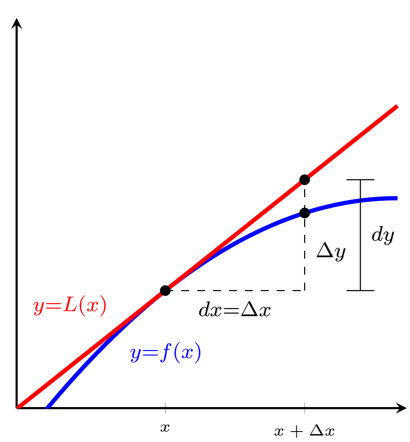
$$dy = f'(x)dx$$

The idea here is that small changes in the input of the function (represented by dx) can have large changes in the output (dy), and those output changes are related to the input changes by the derivative.

Geometrically, we have the following interpretation: Let $P(x, f(x))$ and $Q(x + \Delta x, f(x + \Delta x))$ be points on the graph, and set $dx = \Delta x$. Then

$$\Delta y = f(x + \Delta x) - f(x)$$

And, as we see in the figure below, $\Delta y \approx dy$ becomes a better approximation as $dx = \Delta x$ becomes smaller. Δy is sometimes called the **propogated error** or **error in measurement**.



Remark. If $dx \neq 0$, we can divide both sides to get $\frac{dy}{dx} = f'(x)$, so it turns out this Leibniz notation $\frac{dy}{dx}$ is not just random notation, but rather suggests something about the slope of a function at some infinitesimal distance dx away from the point x.

Example 2.8.3. Find the differential dy given $y = f(x) = \sqrt{x^2+1}$.
We first take the derivative of f.

$$f'(x) = \frac{1}{2}(x^2+1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}}$$

Since $dy = f'(x)\,dx$, we have

$$dy = f'(x)\,dx = \frac{x}{\sqrt{x^2+1}}\,dx.$$

The differentials behave in the ways you might expect.

Theorem 2.8.4 (Product Rule for Differentials)**.** *Let f and g be differentiable functions, and let $y = f(x)g(x)$. Then*

$$dy = g(x)\,df + f(x)\,dg.$$

Proof. By definition, $df = f'(x)\,dx$ and $dg = g'(x)\,dx$. So,

$$\begin{aligned} dy &= y'\,dx \\ &= [f'(x)g(x) + f(x)g'(x)]\,dx \\ &= g(x)f'(x)\,dx + f(x)g'(x)\,dx \\ &= g(x)\,df + f(x)\,dg. \end{aligned}$$

□

Theorem 2.8.5 (Quotient Rules for Differentials)**.** *Let f and g be differentiable functions, and let $y = \dfrac{f(x)}{g(x)}$. Then*

$$dy = \frac{g(x)\,df - f(x)\,dg}{[g(x)]^2}.$$

Proof. By definition, $df = f'(x)\,dx$ and $dg = g'(x)\,dx$. So,

$$\begin{aligned} dy &= y'\,dx \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}\,dx \\ &= \frac{g(x)f'(x)\,dx - f(x)g'(x)\,dx}{[g(x)]^2} \\ &= \frac{g(x)\,df - f(x)\,dg}{[g(x)]^2}. \end{aligned}$$

□

Example 2.8.6. Compute the differential dw given $w = x^{15}\cos(2x)$.
Setting $f(x) = x^{15}$ and $g(x) = \cos(2x)$, we have $w = f(x)g(x)$. A quick calculation of the differentials of f and g shows that

$$\begin{aligned} df &= 15x^{14}\,dx \\ dg &= -2\sin(2x)\,dx. \end{aligned}$$

So, by the product rule above,

$$\begin{aligned} dw &= g(x)\,df + f(x)\,dg \\ &= 15x^{14}\cos(2x)\,dx - 2x^{15}\sin(2x)\,dx \\ &= \left[15x^{14}\cos(2x) - 2x^{15}\sin(2x)\right]\,dx. \end{aligned}$$

These differentials will be immediately useful to us in error analysis.

Definition. Given a quantity Q and a measured error ΔQ, the **relative error** is given by $\dfrac{\Delta Q}{Q}$. If this fraction is expressed as a percentage, we call it the **percentage error**.

Example 2.8.7. The radius of a ball bearing is measured to be 0.7 inch. If the maximum possible error in measurement is 0.01 inch, estimate the largest possible relative error and percentage error in the volume V of the bearing. Recall that the volume of a sphere is given by $V = \frac{4}{3}\pi r^3$, where r is the radius of the bearing. We have that the corresponding error in the calcuated value of V is approximated by the differential

$$dV = 4\pi r^2\,dr$$

When $r = 0.7\,\text{in}$ and $dr = 0.01\,\text{in}$,

$$\begin{aligned} dV &= 4\pi r^2\,dr \\ &= 4\pi(0.7)^2(0.01) \\ &= 0.0616\,\text{in}^3. \end{aligned}$$

The maximum relative error is thus given by

$$\begin{aligned} \frac{\Delta V}{V} \approx \frac{dV}{V} &= \frac{4\pi r^2\,dr}{\frac{4}{3}\pi r^3} \\ &= \frac{3\,dr}{r} \\ &= \frac{3(0.01)}{0.7} \\ &= 0.0429 \text{ or } 4.29\%. \end{aligned}$$

Example 2.8.8. The measurements of the base and altitude of a triangle are 36 centimeters and 50 centimeters, respectively. The possible error in measurement is 0.25 cm. Use differentials to approximate the relative error in computing the area of the triangle.

Recall that the area of a triangle is given by $A = \frac{1}{2}bh$, where b is the base measurement and h is the height/altitude. Since the two measurements, b and h, are independent of one another, the differentials will satisfy the product rule. Hence

$$dA = \frac{1}{2}h\,db + \frac{1}{2}b\,dh.$$

At worst, our measurements were off by a full $0.25\,cm$, so we have $db = dh = 0.25\,\text{cm}$. Thus, our maximum possible error in measurement of the area is

$$\begin{aligned} dA &= \frac{1}{2}h\,db + \frac{1}{2}b\,dh \\ &= \frac{1}{2}(36)(0.25) + \frac{1}{2}(50)(0.25) \\ &= 10.75\,\text{cm}^2. \end{aligned}$$

The maximum relative error is thus given by

$$\begin{aligned} \frac{\Delta A}{A} \approx \frac{dA}{A} &= \frac{\frac{1}{2}h\,db + \frac{1}{2}b\,dh}{\frac{1}{2}bh} \\ &= \frac{db}{b} + \frac{dh}{h} \\ &= \frac{0.25}{36} + \frac{0.25}{50} \\ &\approx 0.0119 \text{ or } 1.19\%. \end{aligned}$$

3 Inverse Functions

3.1 Exponential Functions

Definition. The **exponential function** f **with base** a is

$$f(x) = a^x,$$

where $a > 0$ and $a \neq 1$.

Question 1: What is the domain of an exponential function $f(x) = a^x$?

Well, certainly, if $x = n$ is a nonnegative integer, then

$$a^n = \underbrace{aa \cdots a}_{n} \quad \text{and} \quad a^{-n} = \frac{1}{a^n},$$

so the exponential function is defined on the integers. If $x = \frac{p}{q}$ is a rational number (here p, q are integers), then

$$a^{p/q} = \sqrt[q]{a^p},$$

so exponential functions are even defined on the rational numbers. But what does a^x mean if x is irrational?

Fact. If t is an irrational number, then t can be approximated to arbitrary precision by a rational number r.

Indeed, this makes sense if we consider decimal expansions. For example, $\pi = 3.1415926\ldots$ is an irrational number. However, 3, 3.1, 3.14, 3.141, 3.1415, etc. are all rational numbers and are better and better approximations of π, so it follows that a^3, $a^{3.1}$, $a^{3.14}$, $a^{3.141}$, $a^{3.1415}$, etc. are all defined and are more accurate approximations of a^π.

Proposition 3.1.1. *If t is any real number and $a > 0$ with $a \neq 1$, then*

$$a^t = \lim_{r \to t} a^r \quad \textit{where } r \textit{ is rational.}$$

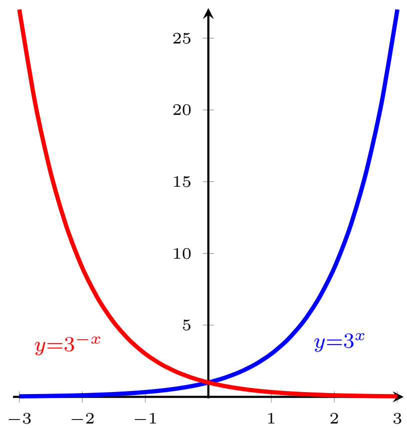
So, all of this work tells us that

Answer 1: The domain of an exponential function is $(-\infty, \infty)$.

Question 2: What is the range of an exponential function $f(x) = a^x$?

We'll go ahead and motivate that with the following:

Example 3.1.2. Below are the graphs of $f(x) = 3^x$ and $g(x) = \left(\frac{1}{3}\right)^x$.



These graphs lead to the following result:

Proposition 3.1.3. *Let $f(x) = a^x$ be an exponential function. If $a > 1$,*

$$\lim_{x\to-\infty} a^x = 0 \quad \text{and} \quad \lim_{x\to\infty} a^x = \infty.$$

If $0 < a < 1$,

$$\lim_{x\to-\infty} a^x = \infty \quad \text{and} \quad \lim_{x\to\infty} a^x = 0.$$

Since $a > 0$, we have that $a^x > 0$ for any real number x, so

Answer 2: The range of $f(x) = a^x$ is $(0, \infty)$.

Question 3: Does an exponential function $f(x) = a^x$ have any vertical or horizontal asymptotes?

Answer 3: Since $f(x)$ is defined on $(-\infty, \infty)$, it doesn't have any vertical asymptotes. Proposition 3.1.3 tells us that $f(x)$ has a horizontal asymptote at $y = 0$.

Theorem 3.1.4. *Let $f(x) = a^x$ be an exponential function. Then f is continuous with domain $(-\infty, \infty)$ and range $(0, \infty)$. Moreover, if a, b, x, y are real numbers and $a, b > 0$, then*

$$\begin{aligned} a^{x+y} &= a^x a^y, \\ a^{x-y} &= \frac{a^x}{a^y}, \\ (a^x)^y &= a^{xy}, \quad \text{and} \\ (ab)^x &= a^x b^x. \end{aligned}$$

Remark. This theorem just tells us that all of the properties we like our exponents to have is true for any real-valued exponent.

Example 3.1.5. Evaluate the limit $\lim_{x\to\infty} \frac{2}{5^x}$.

Notice that we can rewrite this as $\frac{2}{5^x} = 2\left(\frac{1}{5}\right)^x$, so by the constant multiple property of limits and Proposition 3.1.3, we have

$$\lim_{x\to\infty} \frac{2}{5^x} = \lim_{x\to\infty} 2\left(\frac{1}{5}\right)^x = 2\left(\lim_{x\to\infty}\left(\frac{1}{5}\right)^x\right) = 2(0) = 0.$$

Example 3.1.6. Evaluate the limit $\lim_{x\to\infty} \frac{2-5^x}{2+(3)5^x}$.

Since $5^x \neq 0$, we have

$$\begin{aligned}\lim_{x\to\infty} \frac{2-5^x}{2+(3)5^x} &= \lim_{x\to\infty} \frac{5^x\left(\frac{2}{5^x}-1\right)}{5^x\left(\frac{2}{5^x}+3\right)} \\ &= \lim_{x\to\infty} \frac{\frac{2}{5^x}-1}{\frac{2}{5^x}+3} \\ &= \frac{(0)-1}{(0)+3} = -\frac{1}{3}.\end{aligned}$$

It turns out that there is one base that seems to be most convenient for the purposes of calculus.

Definition. The **natural base**, e, is defined to be

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{k\to 0}(1+k)^{1/k}$$

and the function $f(x) = e^x$ is the **natural exponential function** (although often we will just call it "*the* exponential function").

Remark. e is irrational (and transcendental) and $e \approx 2.71828\ldots$.

Example 3.1.7. Evaluate the limit $\lim_{x\to-\infty} e^{7x}\sin(x)$.

Recall that sine has a bounded range of $[-1, 1]$. So, for all x

$$\begin{aligned}-1 &\leq \sin(x) \leq 1 \\ -e^{7x} &\leq e^{7x}\sin(x) \leq e^{7x}.\end{aligned}$$

Since

$$\lim_{x\to-\infty} -e^{7x} = \lim_{x\to-\infty} e^{7x} = 0,$$

then by the Squeeze Theorem,

$$\lim_{x\to-\infty} e^{7x}\sin(x) = 0.$$

Despite seeming contrived, e does arise rather naturally, and its history can be traced back to Jacob Bernoulli in 1685 (22 years before the birth of Leonhard Euler). Recall that the amount A of money in a bank account with principal balance P and annual interest rate r compounded n times per year for t years can be represented by the equation

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Bernoulli was looking at this equation and trying to figure out what would happen if you kept increasing the number of times compounding per year until you were "continuously compounding" the balance. This basically amounts to taking the limit as $n \to \infty$.

$$\begin{aligned}\lim_{n\to\infty} A &= \lim_{n\to\infty} P\left(1 + \frac{r}{n}\right)^{nt} \\ &= \lim_{n\to\infty} P\left(1 + \frac{1}{n/r}\right)^{(n/r)rt}\end{aligned}$$

letting $k = \frac{n}{r}$, we get $k \to \infty$ as $n \to \infty$, so

$$\begin{aligned}&= \lim_{k\to\infty} P\left(1 + \frac{1}{k}\right)^{krt} \\ &= \lim_{k\to\infty} P\left[\left(1 + \frac{1}{k}\right)^{k}\right]^{rt} \\ &= P\left[\lim_{k\to\infty}\left(1 + \frac{1}{k}\right)^{k}\right]^{rt} \\ &= Pe^{rt}.\end{aligned}$$

What Bernoulli noticed was, regardless of the interest rate or the amount of time in the account, this always resulted in the same constant base for an exponential function.

Why is this so important in calculus? It's because of the derivative of the function $f(x) = e^x$. Although we won't fully get to it until Section 3.3, e^x is actually the unique exponential function whose derivative at $x = 0$ is 1. The following argument shows that this is quite plausible:

By definition of e, we have that when h is very small, $e \approx (1+h)^{1/h}$, hence $e^h \approx \left[(1+h)^{1/h}\right]^h = 1+h$, hence $e^h - 1 \approx h$. Thus, for $f(x) = e^x$,

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \frac{e^h - 1}{h} \approx \frac{h}{h} = 1.$$

3.2 Inverse Functions and Logarithms

As with most operations in mathematics, we often like to be able to "undo" them if possible, and this is done with inverse operations: addition and subtraction are inverse operations, multiplication and division are inverse operations. The same is true with functions - we would like to be able to find the inverse operation if possible.

Definition. A function f is called a **one-to-one function** if it never takes on the same value twice; i.e., if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Graphically, a one-to-one function is one that passes the following

Proposition 3.2.1 (Horizontal Line Test)**.** *A function is one-to-one if and only if no horizontal line intersects its graph more than once.*

Example 3.2.2.

a. Is $f(x) = x^3$ one-to-one? Why or why not?

b. Is $f(x) = x^2$ one-to-one? Why or why not?

a. Yes, because two different numbers cannot have the same cube.

b. No, because $f(-2) = f(2) = 4$.

Definition. Suppose f is a one-to-one function with domain A and range B. Then the **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y$$

for any y in B.

Remark. Given f, the inverse function f^{-1} is the unique function satisfying

$$f^{-1} \circ f(x) = x.$$
$$f \circ f^{-1}(y) = y.$$

Remark. The -1 in the notation "f^{-1}" is *not* an exponent, because functions operate by composition, not by multiplication. The notation was created to align with the fact that, given a nonzero number x, multiplication by x^{-1} (aka division by x) is an inverse operation for multiplication. If you want to put a -1 exponent on $f(x)$ to mean $\frac{1}{f(x)}$ use $[f(x)]^{-1}$.

Example 3.2.3. Given $f(x) = x^5$, find the inverse f^{-1} and verify that it is the inverse.
We claim that $f^{-1}(x) = \sqrt[5]{x}$. Indeed, letting $y = x^5$, we have

$$f^{-1} \circ f(x) = f^{-1}(x^5) = \sqrt[5]{x^5} = x$$
$$f \circ f^{-1}(y) = f(\sqrt[5]{y}) = f(\sqrt[5]{x^5}) = f(x) = x^5 = y.$$

so f^{-1} is the inverse of f.

Remark. Since we usually like x to be the independent variable and y the dependent variable, we will define both f and f^{-1} in terms of x, and just keep track of what the domain and range are.

Example 3.2.4. Suppose we know that $f : A \to B$ is a function with domain A and range B with the following values:

$$f(1) = 7, \qquad f(3) = 5, \qquad f(4) = 2.$$

Find $f^{-1}(5)$, $f^{-1}(2)$, and $f^{-1}(7)$.

$$\begin{aligned} f^{-1}(5) = 3 \quad &\text{because} \quad f(3) = 5, \\ f^{-1}(2) = 4 \quad &\text{because} \quad f(4) = 2, \\ f^{-1}(7) = 1 \quad &\text{because} \quad f(1) = 7. \end{aligned}$$

Remark. The procedure below doesn't work in full generality, but only for sufficiently nice functions - there is a brief discussion on this in Section 0.1.4.

Procedure for finding inverses of a one-to-one function f:

1. Replace $f(x)$ with a simpler symbol (might I suggest the letter y?).
2. Switch the roles of x and y in the equation.
3. Solve the above equation for y.
4. Replace the symbol y with $f^{-1}(x)$.

Example 3.2.5. Find the inverse of $f(x) = 2x^3 - 1$. Graph both f and f^{-1}. Step 1 says to write

$$y = 2x^3 - 1.$$

Step 2 says to interchange y and x, writing this as

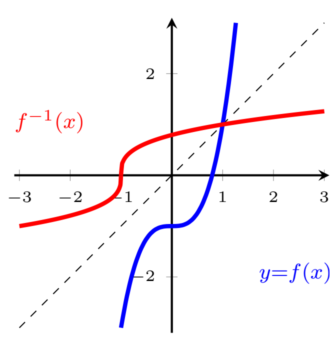
$$x = 2y^3 - 1.$$

Step 3 says to solve for y in terms of x,

$$\begin{aligned} x &= 2y^3 - 1 \\ x + 1 &= 2y^3 \\ \frac{x+1}{2} &= y^3 \\ \Rightarrow \quad y &= \sqrt[3]{\frac{x+1}{2}}. \end{aligned}$$

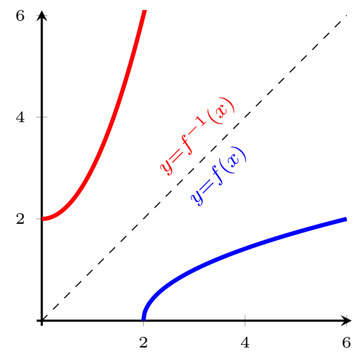
Step 4 says to rewrite y as $f^{-1}(x)$, so we get

$$f^{-1}(x) = \sqrt[3]{\frac{x+1}{2}}$$



What we notice from the previous example is that the graph of a function and its inverse are reflections across the line $y = x$.

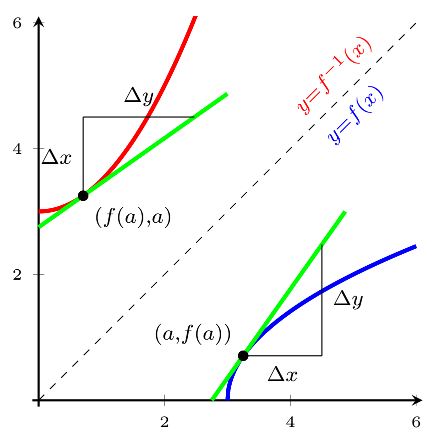
Example 3.2.6. Sketch a graph of $f(x) = \sqrt{x-2}$ and the inverse f^{-1}.



Indeed, if we went through the steps of solving for f^{-1}, we would find that $f^{-1}(x) = x^2 + 2$. But notice that because f has domain $[2, \infty)$ and range $[0, \infty)$, the domain of f^{-1} is only $[0, \infty)$, so we only plot the right half of the function $y = x^2 + 2$.

3.2.1 Calculus of Inverse Functions

From what we know about continuous functions, we have that, if f is continuous and one-to-one, then its inverse should be as well (and indeed it is). What's more, if f is differentiable at a point c and $f'(a) \neq 0$, then we expect that is f^{-1} is also differentiable at $f(a)$.



The picture above even seems to suggest that there is a correspondence between the slope of the tangent line of f at a with the slope of the tangent line of f^{-1} at $f(a)$. The slope of the tangent line of f at a is $m = \frac{\Delta y}{\Delta x}$ and the slope of the tangent line of f^{-1} at $f(a)$ is $\frac{\Delta x}{\Delta y} = \frac{1}{m}$. Indeed, this is always true, and it is shown in the following theorem.

Theorem 3.2.7 (Inverse Function Theorem)**.** *Suppose f is a one-to-one differentiable function with inverse f^{-1} and suppose that $f(a) = b$. If $f'(a) \neq 0$, then*

$$\left(f^{-1}\right)'(b) = \frac{1}{f'(f^{-1}(b))}.$$

Proof. Let $y = f(x)$ and $b = f(a)$, so that $f^{-1}(y) = x$ and $f^{-1}(b) = a$. By continuity, as $y \to b$, then $f^{-1}(y) = x \to f^{-1}(b) = a$.

$$\begin{aligned}\left(f^{-1}\right)'(b) &= \lim_{y \to b} \frac{f^{-1}(y) - f^{-1}(b)}{y - b} \\ &= \lim_{x \to a} \frac{x - a}{f(x) - f(a)} \\ &= \lim_{x \to a} \frac{1}{\frac{f(x)-f(a)}{x-a}} \\ &= \frac{1}{\lim_{x \to a} \frac{f(x)-f(a)}{x-a}} \\ &= \frac{1}{f'(a)} \\ &= \frac{1}{f'(f^{-1}(b))}.\end{aligned}$$

□

Why is this useful? Well, it allows us to find the derivative of the inverse at a point *without having to first calculate the inverse function.*

Example 3.2.8. If $f(x) = \sin(x) + 3x + 2$, find $\left(f^{-1}\right)'(2)$.
By plotting $y = f(x)$, we see that f is one-to-one. Now by inspection, we see that $f(0) = 2$, so $f^{-1}(2) = 0$. We also have that $f'(x) = \cos(x) + 3$. Thus, by the inverse function theorem,

$$\begin{aligned}\left(f^{-1}\right)'(2) &= \frac{1}{f'(f^{-1}(2))} \\ &= \frac{1}{f'(0)} \\ &= \frac{1}{\cos(0) + 3} \\ &= \frac{1}{4}.\end{aligned}$$

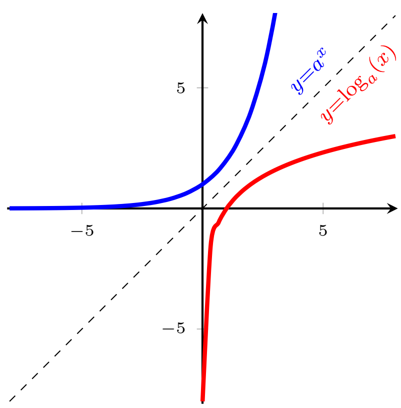
Example 3.2.9. If $f(x) = x^5 + x^4 + x^3 + x^2 + x + 1$, find $\left(f^{-1}\right)'(6)$.
By plotting $y = f(x)$, we see that f is one-to-one. Now, by inspection, we see that $f(1) = 6$, so $f^{-1}(6) = 0$. We also have that $f'(x) = 5x^4 + 4x^3 + 3x^2 + 2x + 1$. Thus, by the inverse function theorem,

$$\begin{aligned}\left(f^{-1}\right)'(6) &= \frac{1}{f'(f^{-1}(6))} \\ &= \frac{1}{f'(1)} \\ &= \frac{1}{5(1)^4 + 4(1)^3 + 3(1)^2 + 2(1) + 1} \\ &= \frac{1}{15}.\end{aligned}$$

3.2.2 Logarithms

When our function is an exponential function (which passes the horizontal line test, and thus has an inverse), we actually give a special name to it.

Definition. If $f(x) = a^x$, the inverse of f is the **logarithmic function with base** a. Using notation as before, if $y = f(x) = a^x$, then $x = \log_a(y)$



By reflecting the graph of an $y = a^x$ as above, we see that

Proposition 3.2.10. *$\log_a(x)$ has domain $(0, \infty)$ and range $[0, \infty)$. We also see that*

$$\lim_{x \to 0} \log_a(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} \log_a(x) = \infty,$$

so $\log_a(x)$ has a vertical asymptote at $x = 0$ and no horizontal asymptotes.

Remark. We will usually omit parentheses when the argument is obvious, we will usually write $\log_a x := \log_a(x)$.

Proposition 3.2.11 (Properties of Logarithms)**.** *We have the following properties, which follow from the definition of the logarithm.*

1. $\log_a a^x =$ x*, for all real* x*.*
2. $a^{\log_a x} =$ x*, for all* $x > 0$*.*
3. $\log_a(xy) =$ $\log_a x + \log_a y$*, for* $x, y > 0$*.*
4. $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$*, for* $x, y > 0$*.*
5. $\log_a x^r = r\log_a x$*, for* $x > 0$ *and any real number* r*.*
6. $\log_a = 0$*.*

Just as e was the special exponential function base,

Definition. The **natural logarithm** is the logarithm with base e and we write $\ln x = \log_e x$.

Remark. In higher levels of mathematics, we often write $\log x$ to mean the natural logarithm, and not the "common logarithm" which you may have seen before. In particular, WolframAlpha does this.

Proposition 3.2.12 (Logarithmic Change of Base Formula)**.** *For any positive real number* a *with* $a \neq 1$*,*

$$\log_a = \frac{\ln x}{\ln a}.$$

Proof. Let $y = \log_a x$. By definition, this means that $a^y = x$. Since logarithmic functions are one-to-one, we can take a natural logarithm of both sides, which gives us

$$y \ln a = \ln x.$$

Rearranging this equation, we get

$$\log_a x = y = \frac{\ln x}{\ln a}.$$

□

Just about every scientific calculator on the planet has a natural logarithm key, but not every calculator has a key for logarithms of any base. This change of base formula is useful for the latter.

3.3 Derivatives of Logarithmic and Exponential Functions

Theorem 3.3.1. *The function* $f(x) = \log_a x$ *is differentiable, and*

$$f'(x) = \frac{1}{x \ln a}.$$

Proof. Since $f^{-1}(x) = a^x$ is differentiable and $(f^{-1})'(x) \neq 0$ for any x, we have that $f'(x)$ exists for every x by the Inverse Function Theorem, thus f is differentiable everywhere. To compute the derivative explicitly, we appeal to the definition of the derivative.

$$\begin{aligned}
f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{\log_a(x+h) - \log_a(x)}{h} \\
&= \lim_{h\to 0} \frac{\log_a\left(\frac{x+h}{x}\right)}{h} && \text{(Logarithm Property 4)} \\
&= \lim_{h\to 0} \frac{1}{h} \log_a\left(\frac{x+h}{x}\right) \\
&= \lim_{h\to 0} \frac{1}{h} \log_a\left(1 + \frac{h}{x}\right) \\
&= \lim_{h\to 0} \frac{1}{x}\frac{x}{h} \log_a\left(1 + \frac{h}{x}\right) \\
&= \frac{1}{x} \lim_{h\to 0} \frac{x}{h} \log_a\left(1 + \frac{h}{x}\right) && \text{(Limit Law \textbf{3.})} \\
&= \frac{1}{x} \lim_{h\to 0} \log_a\left(1 + \frac{h}{x}\right)^{x/h} && \text{(Logarithm Property 5)} \\
&= \frac{1}{x} \log_a\left[\lim_{h\to 0}\left(1 + \frac{h}{x}\right)^{x/h}\right] && \text{($\log_a x$ continuous; Theorem 1.5.7)}
\end{aligned}$$

setting $k = \dfrac{h}{x}$ and noting that as $h \to 0$, $k \to 0$ also, we have

$$\begin{aligned}
&= \frac{1}{x} \log_a\left[\lim_{k\to 0}(1+k)^{1/k}\right] \\
&= \frac{1}{x} \log_a e && \text{(definition of e)} \\
&= \frac{1}{x} \frac{\ln e}{\ln a} && \text{(Change of Base Formula for Logarithms)} \\
&= \frac{1}{x \ln a}.
\end{aligned}$$

□

Corollary 3.3.2. $\dfrac{d}{dx}[\ln x] = \dfrac{1}{x}.$

Example 3.3.3. Given $y = \log_{37} x$, find $\frac{dy}{dx}$.

By Theorem 3.3.1,

$$\frac{dy}{dx} = \frac{1}{x \ln 37}.$$

Example 3.3.4. Find $f'(x)$ where $f(x) = \log_2(x^2 + \cos x)$

Notice that $f(x) = g(h(x))$, where $g(x) = \log_2(x)$ and $h(x) = x^2 + \cos x$. We have that

$$g'(x) = \frac{1}{x \ln 2} \quad \text{and} \quad h'(x) = 2x - \sin x,$$

so applying the chain rule, we get

$$f'(x) = g'(h(x)) \cdot h'(x) = \frac{1}{(x^2 + \cos x)\ln 2} \cdot (2x - \sin x) = \frac{2x - \sin x}{(x^2 + \cos x)\ln 2}.$$

Example 3.3.5. Differentiate $f(x) = x^{13} \ln x$.

Applying the product rule, we have

$$f'(x) = 13x^{12} \ln x + x^{13} \cdot \frac{1}{x} = 13x^2 \ln x + x^{12}.$$

Example 3.3.6. Find $\frac{dg}{du}$ where $g(u) = \frac{1+u}{1+\ln u}$.

Applying the quotient rule, we have

$$\frac{dg}{du} = \frac{1(1 + \ln u) - (1+u)\frac{1}{u}}{(1 + \ln u)^2} = \frac{\ln u - \frac{1}{u}}{(1 + \ln u)^2} = \frac{u \ln u - 1}{u(1 + \ln u)^2}.$$

Example 3.3.7. Find y' for $y = \ln |x|$.

Following from the definition of the absolute value, we have

$$\ln |x| = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0. \end{cases}$$

So, if $x > 0$, we have

$$\frac{d}{dx}[\ln |x|] = \frac{d}{dx} \ln x = \frac{1}{x},$$

and if $x < 0$, we have

$$\frac{d}{dx}[\ln |x|] = \frac{d}{dx} \ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

Thus $y' = \frac{1}{x}$ for all $x \neq 0$.

3.3.1 Logarithmic Differentiation

The following example gives a procedure, known as **logarithmic differentiation**, for finding derivatives that may involve copious amounts of product/quotient rules, as well as derivatives of exponential functions (which is something we don't have yet).

Example 3.3.8. Differentiate $y = \dfrac{(x^2-7)\sqrt{x^{5/2}+x}}{(2x+9)^{750}}$.

Since logarithms are injective, we take a logarithm of both sides and use the properties of logarithms to simplify the expression:

$$\begin{aligned}\ln y &= \ln(x^2-7) + \ln\sqrt{x^{5/2}+x} - \ln(2x+9)^{750}\\ &= \ln(x^2-7) + \frac{1}{2}\ln(x^{5/2}+x) - 750\ln(2x+9).\end{aligned}$$

Differentiating implicitly with respect to x gives us

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2-7} + \frac{1}{2}\cdot\frac{\frac{5}{2}x+1}{x^{5/2}+x} - 750\cdot\frac{2}{2x+9},$$

and solving for $\frac{dy}{dx}$, we get

$$\begin{aligned}\frac{dy}{dx} &= y\left(\frac{2x}{x^2-7} + \frac{1}{2}\cdot\frac{\frac{5}{2}x+1}{x^{5/2}+x} - 750\cdot\frac{2}{2x+9}\right)\\ &= \frac{(x^2-7)\sqrt{x^{5/2}+x}}{(2x+9)^{750}}\left(\frac{2x}{x^2-7} + \frac{1}{2}\cdot\frac{\frac{5}{2}x+1}{x^{5/2}+x} - 750\cdot\frac{2}{2x+9}\right).\end{aligned}$$

Procedure for logarithmic differentiation:

1. Take the natural logarithm of both sides of the equation $y = f(x)$.
2. Simplify with the properties of logarithms.
3. Implicitly differentiate with respect to x.
4. Solve the above equation for y'.

Logarithmic differentiation allows us to prove a more general form of the power rule.

Proposition 3.3.9 (Power Rule)**.** *If r is any real number and $f(x) = x^r$, then*

$$f'(x) = rx^{r-1}.$$

Proof. If $x < 0$, it may be that $\ln x^r$ is undefined. Fortunately, the result of Example 3.3.7 tells us that we can use $\ln|x^r|$ instead. Recall also a property of absolute values that $|x^r| = |x|^r$. So, setting $y = x^r$, and using logarithmic differentiation, we have

$$\ln|y| = \ln|x^r| = \ln|x|^r = r\ln|x|.$$

By Example 3.3.7, differentiating implicitly with respect to x gets us

$$\begin{aligned}\frac{y'}{y} &= \frac{r}{x}\\ \Rightarrow \quad y' &= y\frac{r}{x} = \frac{rx^r}{x} = rx^{r-1}.\end{aligned}$$

□

3.3.2 Derivatives of Exponentials

Theorem 3.3.10. *For $a > 0$, the function $f(x) = a^x$ is differentiable and*

$$\frac{d}{dx}[a^x] = a^x \ln a.$$

Proof. Since $f(x) = \log_a x$ is differentiable, the inverse $f(x) = a^x$ is differentiable.

Set $y = a^x$. This is equivalent to $\log_a y = x$. Implicitly differentiating this equation with respect to x, we get

$$\begin{aligned} \frac{1}{y \ln a} \frac{dy}{dx} &= 1 \\ \Rightarrow \quad \frac{dy}{dx} &= y \ln a = a^x \ln a. \end{aligned}$$

□

Corollary 3.3.11. $\frac{d}{dx}[e^x] = e^x$.

Example 3.3.12. Find $\frac{df}{dx}$ where $f(x) = 3^{x^5}$.

Notice that $f(x) = g(h(x))$ where $g(x) = 2^x$ and $h(x) = x^5$. We have that

$$g'(x) = 2^x \ln 2 \quad \text{and} \quad h'(x) = 5x^4,$$

so applying the chain rule, we get

$$f'(x) = g'(h(x)) \cdot h'(x) = (2^{x^5} \ln 2)(5x^4).$$

Example 3.3.13. Find y'' given $y = \cos(e^x)$.

Applying a chain rule, we get

$$y' = -\sin(e^x)e^x$$

and applying both a chain and product rule to y', we get

$$y'' = -\cos(e^x)e^{2x} - \sin(e^x)e^x$$

Example 3.3.14. If $f(x) = \ln(3)e^5$, then what is $f'(x)$?

e^5 and $\ln(3)$ are both real numbers, so f is a constant function, and thus $f'(x) = 0$.

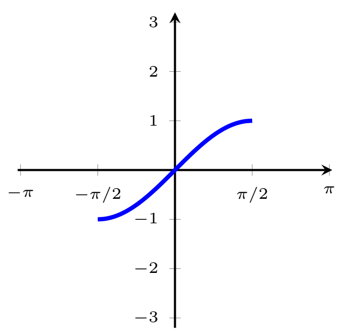
3.5 Inverse Trigonometric Functions

3.5.1 Preliminaries

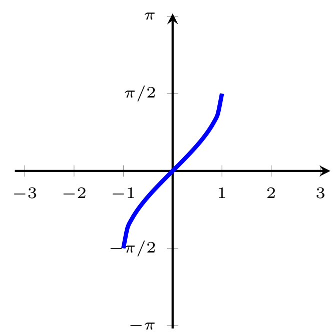
As we've seen before, the sine function $y = \sin x$ is not one-to-one because it fails the horizontal line test. As such, sine is not invertible on its entire domain. However, it *is* one-to-one on the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. So, we can find its inverse.

Definition. The **inverse sine function**, denoted $y = \sin^{-1} x$ or $y = \arcsin x$, is the function with domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ satisfying

$$y = \sin^{-1} x \quad \text{if and only if} \quad \sin y = x.$$



$y = \sin x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

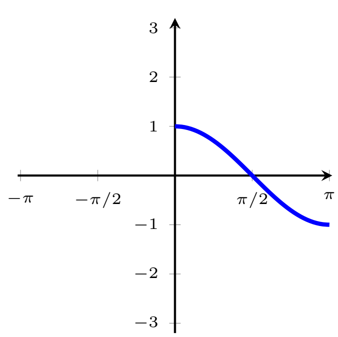


$y = \sin^{-1} x$, $-1 \le x \le 1$

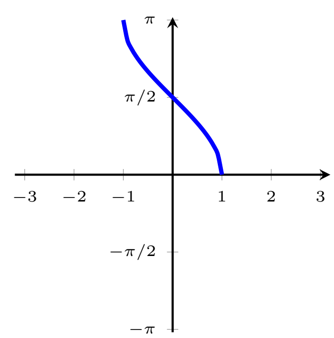
Similarly, the cosine function $y = \cos x$ is not one-to-one because it fails the horizontal line test, so is not invertible on its entire domain. However, we can restrict the domain to $[0, \pi]$ where it is one-to-one and find its inverse.

Definition. The **inverse cosine function**, denoted $y = \cos^{-1} x$ or $y = \arccos x$, is the function with domain $[-1, 1]$ and range $[0, \pi]$ satisfying

$$y = \cos^{-1} x \quad \text{if and only if} \quad \cos y = x.$$



$y = \cos x$, $0 \le x \le \pi$



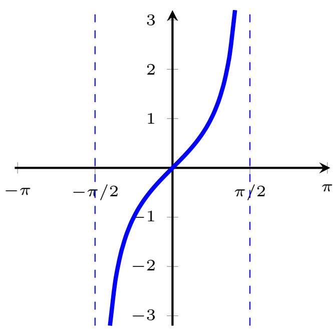
$y = \cos^{-1} x$, $-1 \le x \le 1$

Remark. It would be useful at this juncture to remind the reader that the -1 in $\sin^{-1} x$ and $\cos^{-1} x$ *does not mean* $\frac{1}{\sin x}$ or $\frac{1}{\cos x}$; we already have names for these reciprocal trigonometric functions. Yes, the -1 notation is inconsistent with notation like $\sin^2 x$, but both notations are unfortunately so common that it's nigh impossible to phase out either.

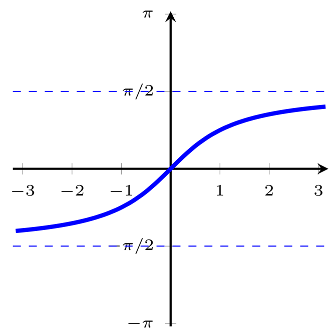
The tangent function $y = \tan x$ is also not one-to-one, because it fails the horizontal line test. As such, it is not invertible on its entire domain, but we can restrict the domain to $(0, \pi)$ where it is one-to-one and invertible.

Definition. The **inverse tangent function**, denoted $y = \tan^{-1} x$ or $y = \arctan x$, is the function with domain $(-\infty, \infty)$ and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfying

$$y = \tan^{-1} x \quad \text{if and only if} \quad \tan y = x.$$



$y = \cos x,\ 0 \le x \le \pi$



$y = \tan^{-1} x,\ -\infty < x < \infty$

Remark. $\tan^{-1} x \neq \dfrac{\sin^{-1} x}{\cos^{-1} x}$!

Because $y = \tan x$ has vertical asymptotes at $x = (2n+1)\pi/2$ for all integers n, the following proposition follows immediately:

Proposition 3.5.1. *With $y = \tan^{-1} x$ as given,*

$$\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2} \qquad \textit{and} \qquad \lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}.$$

The remaining three inverse trigonometric functions can be defined similarly to the three we've already seen.

Definition. The **inverse cosecant function**, denoted by $y = \csc^{-1} x$ or $y = \operatorname{arccsc} x$, is the function with domain $(-\infty, -1] \cup [1, \infty)$ and range $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$ satisfying

$$y = \csc^{-1} x \quad \text{if and only if} \quad \csc y = x.$$

Definition. The **inverse secant function**, denoted by $y = \sec^{-1} x$ or $y = \operatorname{arcsec} x$, is the function with domain $(-\infty, -1] \cup [1, \infty)$ and range $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ satisfying

$$y = \sec^{-1} x \quad \text{if and only if} \quad \sec y = x.$$

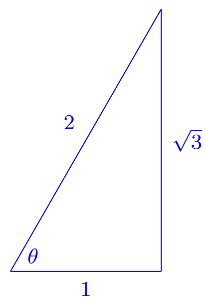
Definition. The **inverse cotangent function**, denoted by $y = \cot^{-1} x$ or $y = \operatorname{arccot} x$, is the function with domain $(-\infty, \infty)$ and range $(0, \pi)$ satisfying

$$y = \cot^{-1} x \quad \text{if and only if} \quad \cot y = x.$$

Remark. The domains for inverse cosecant and inverse secant are not universally agreed upon. The ones we have chosen tend to yield slightly simpler differentiation formulas.

Example 3.5.2. Find $\arctan(\sqrt{3})$ exactly.

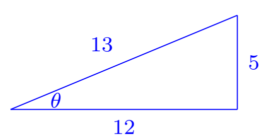
Since $\arctan(\sqrt{3}) = \theta$ is equivalent to $\tan\theta = \sqrt{3} = \frac{\sqrt{3}}{1}$, we have that $\theta = \frac{\pi}{3}$. If you don't remember this, note that we have the following triangle



With this triangle, we've reduced it to finding the θ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ which makes $\sin\theta = \frac{\sqrt{3}}{2}$ and $\cos\theta = \frac{1}{2}$. This angle is, of course, $\frac{\pi}{3}$.

Example 3.5.3. Find $\cos\left(\sin^{-1}\frac{5}{13}\right)$ exactly.

The beauty here is that we don't actually need to know anything about the angle to solve this. Since $\sin^{1}\frac{5}{13} = \theta$ is equivalent $\sin\theta = \frac{5}{13}$, we can set up our triangle again



From this triangle, we see then that

$$\cos\left(\sin^{-1}\frac{5}{12}\right) = \cos\theta = \frac{12}{13}.$$

Example 3.5.4. Find $\sin^{-1}\left(\sin\frac{4\pi}{3}\right)$ exactly.

You may be tempted to write that the answer is $\frac{4\pi}{3}$, but in fact this is outside of the range of $\sin^{-1}$, so it can't be right. Instead, solving it "from the inside out", we get

$$\sin^{-1}\left(\sin\frac{4\pi}{3}\right) = \sin^{-1} -\frac{\sqrt{3}}{2} = -\frac{\pi}{3}.$$

3.5.2 Calculus of Inverse Trigonometric Functions

Theorem 3.5.5. *All six inverse trigonometric functions are differentiable and their derivatives are*

$$\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left[\csc^{-1} x\right] = -\frac{1}{x\sqrt{x^2-1}}$$
$$\frac{d}{dx}\left[\cos^{-1} x\right] = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left[\sec^{-1} x\right] = \frac{1}{x\sqrt{x^2-1}}$$
$$\frac{d}{dx}\left[\tan^{-1} x\right] = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left[\cot^{-1} x\right] = -\frac{1}{1+x^2}$$

Proof. Certainly these functions are differentiable since their inverses are differentiable, so we will just find the derivative of $y = \cos^{-1} x$ and leave the remaining five derivatives as an exercise.

Since $y = \cos^{-1} x$, we have that $\cos y = x$. Implicitly differentiating this equation yields

$$\begin{aligned} \frac{d}{dx}[\cos y] &= \frac{d}{dx}[x] \\ -\sin y \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= -\frac{1}{\sin y}. \end{aligned}$$

Now, by the pythagorean identity, $\sin y = \sqrt{1-\cos^2 y} = \sqrt{1-x^2}$, so

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}.$$

□

Example 3.5.6. Let $y = \arcsin(2x)$. Find $\frac{dy}{dx}$. From Theorem 3.5.5 and the chain rule, it follows very quickly that we have

$$y' = \frac{2}{1+(2x)^2} = \frac{2}{1+4x^2}.$$

Example 3.5.7. Find $f'\left(\frac{1}{2}\right)$ where $f(x) = 3\arccos(x^2)$. Once again, from Theorem 3.5.5 and the chain rule, we have

$$f'(x) = -\frac{6x}{\sqrt{1-(x^2)}} = -\frac{6x}{\sqrt{1-x^4}}.$$

So, when $x = \frac{1}{2}$, we get

$$f'\left(\frac{1}{2}\right) = -\frac{3}{\sqrt{1-1/16}} = -4\sqrt{\frac{3}{5}} \approx -3.09839.$$

Example 3.5.8. Find y'' where $y = \arctan(x)$. From Theorem 3.5.5, we have

$$y' = \frac{1}{1+x^2}.$$

Differentiating with respect to x a second time yields

$$y'' = -\frac{2x}{(1+x^2)^2}.$$

In this author's own experience, the $\arcsin x$ and $\arctan x$ functions tend to appear most often in the real world because $\sin x$ and $\tan x$ are incredibly common.

3.6 Indeterminate Forms and L'Hospital's Rule

3.6.1 0/0 and ∞/∞ Indeterminate Forms

We have seen limits like the following

$$\lim_{x\to\infty} \frac{e^x}{5^x} \quad \text{and} \quad \lim_{x\to 1} \frac{x^2-1}{x-1}.$$

We are unable to simply plug-in the limits as we would end up with "$\frac{\infty}{\infty}$" and "$\frac{0}{0}$", respectively which are both undefined. However, we were able to still find the limits, and what's more, both limits were very different! This reaffirms what we've known - that infinity doesn't quite behave like other real numbers, and division by 0 is equally as ill-behaved.

Definition. Let $f(x)$ and $g(x)$ be functions and consider the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)}.$$

If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then we say this limit is an **indeterminate of type $\frac{0}{0}$.** Similarly, if $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$, then we say that this limit is an **indeterminate form of type $\frac{\infty}{\infty}$.**

For limits of these types, we had various methods of evaluating the limits. For example, in the case of rational functions, we factored and canceled terms. However, such tricks may not work for other "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" indeterminate forms, like

$$\lim_{x\to 0} \frac{\cos x - 1}{2^{x+1}-2} \quad \text{or} \quad \lim_{x\to\infty} \frac{\ln x}{x+7}.$$

Now that we have derivatives in our toolbox, we can make use of the following theorem.

Theorem 3.6.1. *L'Hospital's Rule Suppose f and g are differentiable functions with $g'(x) \neq 0$ near a (except possibly at a). Suppose also that*

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

is an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Proof. We'll prove the *very* special case where we have an indeterminate form of type $\frac{0}{0}$, $f(a) = g(a) = 0$, f' and g' are continuous, and $g'(a) \neq 0$; the proof of the theorem in full generality is much more difficult and can be found in the textbook.

$$\begin{aligned}
\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} && \text{(since } f(a)=g(a)=0\text{)}\\
&= \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}\cdot\frac{\frac{1}{x-a}}{\frac{1}{x-a}}\\
&= \lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}\\
&= \frac{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}}\\
&= \frac{f'(a)}{g'(a)}\\
&= \frac{\lim_{x\to a}f'(x)}{\lim_{x\to a}g'(x)} && \text{(since } f',g' \text{ are continuous)}\\
&= \lim_{x\to a}\frac{f'(x)}{g'(x)}.
\end{aligned}$$

□

Remark. L'Hospital's rule applies for one-sided limits and limits at infinity as well.

Example 3.6.2. Evaluate $\lim_{x\to\infty}\frac{\ln x}{x}$.

We first see that this is an indeterminate form of type $\frac{\infty}{\infty}$, so we can apply l'Hospital's rule.

$$\lim_{x\to\infty}\frac{\ln x}{x}\overset{LH}{=}\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=\lim_{x\to\infty}\frac{1}{x}=0.$$

Remark. Notationally, the author likes to use $\overset{LH}{=}$ to indicate when l'Hospital's rule has been applied. This notation is not at all standard.

Example 3.6.3. Evaluate $\lim_{t\to 0}\frac{\sin t}{t}$.

We first see that this is an indeterminate form of type $\frac{0}{0}$, so we can apply l'Hospital's rule.

$$\lim_{x\to 0}\frac{\sin x}{x}\overset{LH}{=}\lim_{x\to 0}\frac{\cos x}{1}=1.$$

Example 3.6.4. Evaluate $\lim_{x\to 1}\frac{\arctan x-\frac{\pi}{4}}{x-1}$.

Recall that $\arctan 1=\frac{\pi}{4}$, so we indeed have an indeterminate form of type $\frac{0}{0}$, and thus we can apply l'Hospital's rule.

$$\lim_{x\to 1}\frac{\arctan x-\frac{\pi}{4}}{x-1}\overset{LH}{=}\lim_{x\to 1}\frac{\frac{1}{1+x^2}}{1}=\frac{1}{2}.$$

Example 3.6.5. Evaluate $\lim_{x \to -\infty} \frac{x^2}{e^{-x}}$.

We first see that this is an indeterminate form of type $\frac{\infty}{\infty}$, so we can apply l'Hospital's rule.

$$\lim_{x \to -\infty} \frac{x^2}{e^{-x}} \stackrel{LH}{=} \lim_{x \to -\infty} \frac{2x}{-e^{-x}}.$$

Once again, this new limit is an indeterminate form of type $\frac{\infty}{\infty}$, so we can apply l'Hospital's rule again.

$$\lim_{x \to -\infty} \frac{2x}{-e^{-x}} \stackrel{LH}{=} \lim_{x \to -\infty} \frac{2}{e^{-x}} = 0.$$

Remark. L'Hospital's rule **can only be applied** to indeterminate forms of types $\frac{0}{0}$ and $\frac{\infty}{\infty}$; for other indeterminate forms (which we'll get to later).

To see why the indeterminate form is an important hypothesis, consider the following (non)example:

Example 3.6.6. Evaluate $\lim_{\theta \to \pi^-} \frac{\sin\theta}{1 - \cos\theta}$ by applying l'Hospital's rule blindly. Find the correct limit without l'Hospital's rule.

Blindly applying l'Hospital's to the following limit,

$$\lim_{\theta \to \pi^-} \frac{\sin\theta}{1 - \cos\theta} \stackrel{LH}{=} \lim_{\theta \to \pi^-} \frac{\cos\theta}{\sin\theta} = -\infty.$$

However, **this is wrong.** To see why, note that $\frac{\sin\theta}{1 - \cos\theta}$ is actually continuous at $\theta = \pi$, so in fact we have

$$\lim_{\theta \to \pi^-} \frac{\sin\theta}{1 - \cos\theta} = \frac{\sin\pi}{1 - \cos\pi} = \frac{0}{1 - (-1)} = 0.$$

3.6.2 $0 \cdot \infty$ and ∞-∞ indeterminate forms

Definition. Let f and g be functions and consider the limit

$$\lim_{x \to a} f(x)g(x).$$

If $f(x) \to 0$ and $g(x) \to \pm\infty$ as $x \to a$, then we say that this limit is an **indeterminate form of type $0 \cdot \infty$.**

To deal with limits of this form, the technique is usually to rewrite $f(x)g(x)$ as either $\frac{f(x)}{1/g(x)}$ (resulting in an indeterminate form of type $\frac{0}{0}$) or $\frac{g(x)}{1/g(x)}$ (resulting in an indeterminate form of type $\frac{\infty}{\infty}$) and then applyin l'Hospital's rule.

Example 3.6.7. Evaluate $\lim_{x\to\infty} e^{-x}\sqrt{x}$.

First we see that $e^{-x} \to 0$ and $\sqrt{x} \to \infty$, so indeed we have an indeterminate form of type $0 \cdot \infty$. Rewriting

$$e^{-x}\sqrt{x} = \frac{\sqrt{x}}{e^x},$$

we now have an indeterminate form of type $\frac{\infty}{\infty}$, so we can apply l'Hospital's rule.

$$\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty} \frac{\sqrt{x}}{e^x} \overset{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{e^x} \lim_{x\to\infty} \frac{1}{2e^x\sqrt{x}} = 0.$$

Definition. Let f and g be functions and consider the limit

$$\lim_{x\to a} f(x) - g(x).$$

If $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$, then we say that have an **indeterminate form of type $\infty - \infty$.**

The technique for solving these limits is to convert the difference into a quotient, often by rationalizing or finding a common denominator, and then applying l'Hospital's rule.

Example 3.6.8. Evalute $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{1-e^{-x}}\right)$.

By rewriting as a fraction with a common denominator, we have that

$$\frac{1}{x} - \frac{1}{1-e^{-x}} = \frac{1-e^{-x}-x}{x-xe^{-x}},$$

which is indeterminate of type $\frac{0}{0}$. So, applying l'Hospital's rule,

$$\lim_{x\to 0^+} \frac{1-e^{-x}-x}{x-xe^{-x}} \overset{LH}{=} \lim_{x\to 0^+} \frac{e^{-x}-1}{1-e^{-x}+xe^{-x}}.$$

This is again an indeterminate form of type $\frac{0}{0}$. So, applying l'Hospital's rule again,

$$\lim_{x\to 0^+} \frac{e^{-x}-1}{1-e^{-x}+xe^{-x}} \overset{LH}{=} \lim_{x\to 0^+} \frac{-e^{-x}}{e^{-x}+e^{-x}-xe^{-x}} = -\frac{1}{2}.$$

3.6.3 0^0, 1^∞, and ∞^0 indeterminate forms

Definition. Let f and g be functions and consider the limit

$$\lim_{x\to a} [f(x)]^{g(x)}.$$

If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then we say that have an **indeterminate form of type 0^0.**

If $f(x) \to 1$ and $g(x) \to \infty$ as $x \to a$, then we say that have an **indeterminate form of type 1^∞.**

If $f(x) \to \infty$ and $g(x) \to 0$ as $x \to a$, then we say that have an **indeterminate form of type ∞^0.**

There are two equivalent techniques to handle these types of indeterminate forms (and in fac. The first is to take a natural logarithm of the limit. That is, let $L = \lim_{x\to a}[f(x)]^{g(x)}$. Then, since the logarithmic functions are continuous,

$$\ln L = \ln\left(\lim_{x\to a}[f(x)]^{g(x)}\right) = \lim_{x\to a}\ln\left([f(x)]^{g(x)}\right) = \lim_{x\to a} g(x)\cdot\ln(f(x)).$$

Once you have solved for this limit on the right (usually by rewriting into "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$" and applying L'Hospital's rule), simply "undo" this natural logarithm by raising the function into the exponent with base e; i.e.

$$L = e^{\ln L} = e^{\lim_{x\to a} g(x)\cdot\ln(f(x))}.$$

The other technique is to appeal to the fact that $[f(x)]^{g(x)} = e^{g(x)\ln f(x)}$, that is,

$$\lim_{x\to a}[f(x)]^{g(x)} = \lim_{x\to a} e^{g(x)\ln(f(x))}.$$

Given that exponential functions are equivalent, this just amounts to solving for the limit of $g(x)\ln(f(x))$, and so the technique is basically equivalent to the first one; either should work.

Example 3.6.9. Evaluate $\lim_{x\to 0^+} x^x$. Recall that, because the natural log is continuous, we have

$$\lim_{x\to a}\ln(f(x)) = \ln\left(\lim_{x\to a} f(x)\right).$$

We see that this limit is an indeterminate form of type 0^0. So, let L be the limit. Taking a natural logarithm of the limit, we get

$$\ln L = \ln\left(\lim_{x\to 0^+} x^x\right) = \lim_{x\to 0^+} x\ln x.$$

This is now indeterminate of type $0\cdot\infty$, so we rewrite $x\ln x = \frac{\ln x}{1/x}$ to put change it to an indeterminate form of type $\frac{\infty}{\infty}$, whence we can apply l'Hospital's Rule.

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x} \overset{LH}{=} \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0.$$

But we're not quite done yet. Notice that we just showed that $\ln L = 0$, but we wanted to find the value of L. From here we deduce that the value of the limit is $L = 1$.

Example 3.6.10. Evaluate $\lim_{x\to\infty}\left(1+\frac{7}{x}\right)^x$.

We see that this is an indeterminate form of type 1^∞. Just as last time, let L be the value of the limit. Taking a natural logarithm of the limit, we get

$$\ln L = \ln\left(\lim_{x\to\infty}\left(1+\frac{7}{x}\right)^x\right) = \lim_{x\to\infty}\ln\left(\left(1+\frac{7}{x}\right)^x\right) = \lim_{x\to\infty} x\ln\left(1+\frac{7}{x}\right).$$

As $x\to\infty$, $\ln\left(1+\frac{7}{x}\right)\to 0$, so we now have an indeterminate form of type $0\cdot\infty$. Rewriting as $\frac{\ln\left(1+\frac{7}{x}\right)}{1/x}$, we get an indeterminate form of type $\frac{\infty}{\infty}$, and can thus apply l'Hospital's rule.

$$\lim_{x\to\infty} x\ln\left(1+\frac{7}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{7}{x}\right)}{1/x} \overset{LH}{=} \lim_{x\to\infty}\frac{\frac{1}{1+7/x}\cdot\frac{-7}{x^2}}{-1/x^2} = \lim_{x\to\infty}\frac{-\frac{7}{x^2+7x}}{-1/x^2} = \lim_{x\to\infty}\frac{7x^2}{x^2+7x} = 7.$$

So now we have that $\ln L = 7$, and thus $L = e^7$.

Example 3.6.11. Evaluate $\lim_{x\to\infty} x^{1/x}$.

We notice that this is indeterminate of type ∞^0, so can rewrite $x^{1/x} = e^{(1/x)\ln x}$. Since exponential functions are continuous, taking the whole limit amounts to taking a limit of the exponent. So, let $M = \lim_{x\to\infty} \frac{1}{x}\ln x$, which we recognize as an indeterminate form of type $0 \cdot \infty$. Rearranging it as $\frac{\ln x}{x}$, which is indeterminate of type $\frac{\infty}{\infty}$, and so we can apply l'Hospital's Rule.

$$M = \lim_{x\to\infty} \frac{1}{x}\ln x = \lim_{x\to\infty} \frac{\ln x}{x} \stackrel{LH}{=} \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

So then our entire limit is

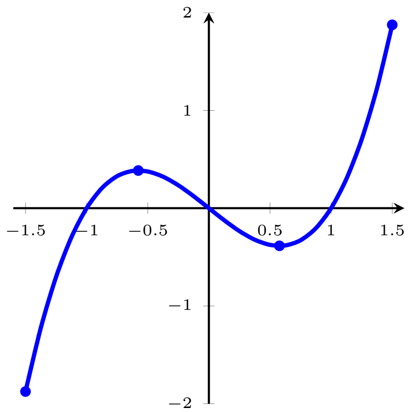
$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{(1/x)\ln x} = e^M = e^0 = 1.$$

4 Applications of Differentiation

4.1 Minimum and Maximum Values

Definition. Let c be a number in the domain D of a function f. We say that $f(c)$ is an **absolute minimum** (or **global minimum**) value if, for all x in D, $f(c) \leq f(x)$. We say that $f(c)$ is an **absolute maximum** (or **global maximum**) value if, for all x in D, $f(c) \geq f(x)$. The minimum and maximum values of f are known as **extreme values** or just **extrema**.

Example 4.1.1. Consider the function $f(x) = x^3 + x$ where $-1.5 \leq x \leq 1.5$. Using a graph of the function, determine the global minimum and maximum values.



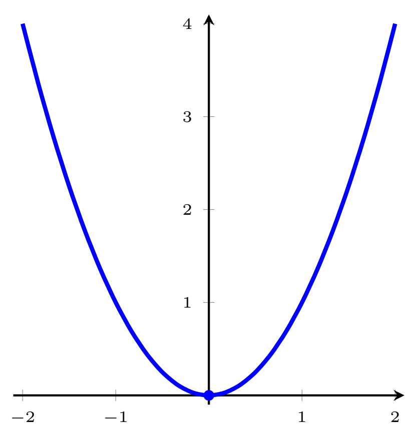
The absolute minimum occurs at $x =$ __-1.5__ and the value is __-1.875__.

The absolute maximum occurs at $x =$ __1.5__ and the value is __1.875__.

In the previous example, the two points $\left(\pm\frac{1}{\sqrt{3}}, \mp\frac{2}{3\sqrt{3}}\right)$ are not global extrema, but would be if we restricted our focus to just points around them.

Definition. Given f and c as in the previous definition, the number $f(c)$ is a **local minimum** (or **relative minimum**) value of f if $f(c) \leq f(x)$ for all x in an open interval containing c. The number $f(c)$ is a **local maximum** (or **relative maximum**) value of f if $f(c) \geq f(x)$ for all x in an open interval containing c. Local maxima/minima are sometimes referred to as **local extrema**.

Example 4.1.2. What is the local minimum value of $f(x) = x^2$? What is the absolute minimum value of $f(x) = x^2$?

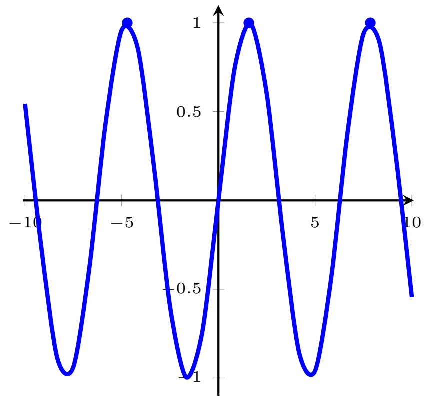


The local minimum occurs at $x =$ __0__ and the value is __0__.

The absolute minimum occurs at $x =$ __0__ and the value is __0__.

This last example indicates to us that local minima/maxima and absolute minima/maxima are not mutually exclusive. Indeed, they may be the same.

Example 4.1.3. What is the local maximum value of $g(x) = \sin x$? What is the absolute maximum value?



The local minimum occurs at $x = \underline{2n\pi + \frac{\pi}{2}}$ and the value is $\underline{1}$.

The absolute minimum occurs at $x = \underline{2n\pi + \frac{\pi}{2}}$ and the value is $\underline{1}$.

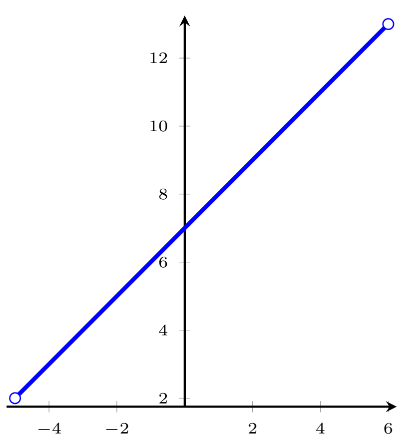
This last example indicates that the local and absolute extrema may be achieved infinitely many times.

As well, it may be that we have several different local maxima/minima (e.g., $y = x\cos x$), or none at all (e.g, $y = 5x + 7$). Similarly, absolute maxima/minima need not exist at all.

The following result tells us when absolute maxima/minima exist.

Theorem 4.1.4 (Extreme Value Theorem)**.** *If f is continuous on a closed interval $[a, b]$, then there are numbers c and d in the interval $[a, b]$ so that f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$.*

Example 4.1.5. To see why the closed interval condition is required, consider the function $f(x) = x+7$ on the open interval $(-5, 6)$.

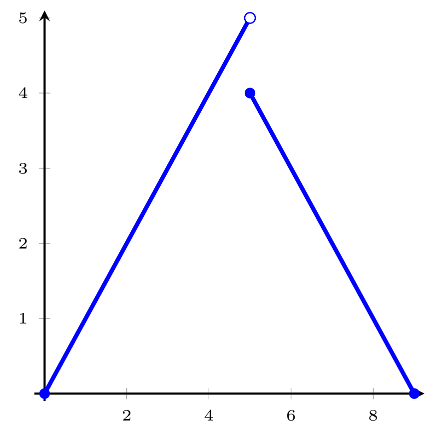


If there were a maximum value, it would be at 13 which occurs at $x = 6$, but 6 is not in the domain, so f does not attain an absolute maximum. By a similar argument, f does not attain an absolute minimum value.

Example 4.1.6. To see why the continuity is required, consider the following piecewise function with domain $[0, 9]$.

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 5 \\ 9 - x & \text{if } 5 \leq x \leq 9 \end{cases}$$

Certainly this function has an absolute minimum value of 0 (occurring at both $x = 0$ and $x = 9$), but it does not have an absolute maximum value.

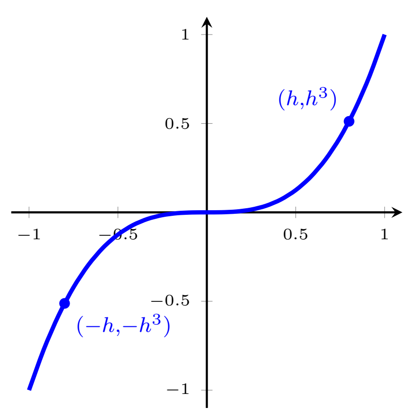


Now notice that local extrema occur at "peaks" and "valleys" on the graph of a function. The tangent lines appear horizontal at these points. Indeed, they are, and the following theorem tells us when local minima/maxima may exist.

Theorem 4.1.7. *If $f(c)$ is a local minimum or maximum and $f'(c)$ exists, then $f'(c) = 0$.*

Remark. It is important to note the logical implication of this theorem. The converse of the statement above is not necessarily true: having a zero derivative at a point is not sufficient to show that the point is a local minimum/maximum.

Example 4.1.8. Consider the function $f(x) = x^3$. Then $f'(x) = 3x^2 = 0$ precisely when $x = 0$. However, notice that 0 is not a local minimum or maximum because $(x - h)^3 < x^3 < (x + h)^3$ for all nonzero h, and in particular, $(-h)^3 < 0 < h^3$.



Although it may not correspond to a local minimum or maximum, we still give a name to x-values where the derivative is 0.

Definition. A **critical number** or **critical point** of a function f is the number c in the domain of f where either $f'(c) = 0$ or $f'(c)$ does not exist.

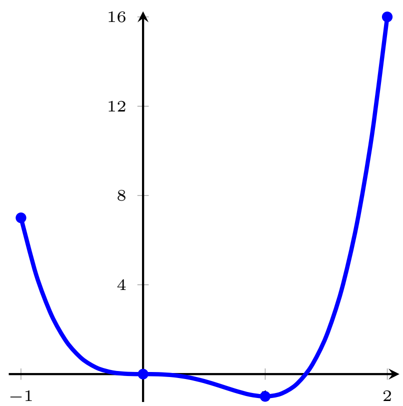
Procedure for finding extrema on a closed interval $[a, b]$:

1. Find the critical points of f in (a, b).
2. Evaluate f at each critical point in (a, b).
3. Evaluate f at each endpoint of the interval; i.e. find $f(a)$ and $f(b)$.
4. The smallest and largest values from the two previous steps are the absolute maximum and minimum, respectively.

Example 4.1.9. Find the extrema of $f(x) = 3x^4 - 4x^3$ on the interval $[-1, 2]$.

Since $f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1)$ we see that we have critical points at $x = 0$ and $x = 1$. We thus create the following table of values.

x	$f(x)$
-1	7
0	0
1	-1
2	16



The absolute minimum occurs at $x =$ __1__ and the value is __-1__.

The absolute minimum occurs at $x =$ __2__ and the value is __16__.

4.2 The Mean Value Theorem

Last time we talked about when local maxima and minima may exist, and we determined that critical points (in particular, zero derivatives) played a crucial roll. The following also gives us a criterion in which to check when critical points may exist.

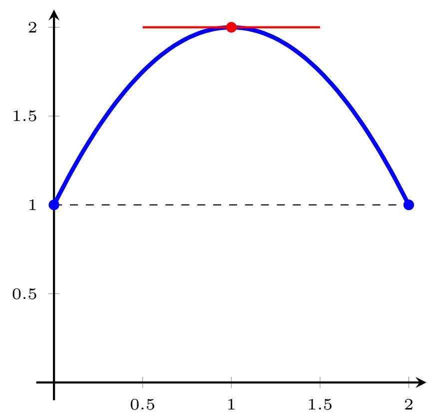
Theorem 4.2.1 (Rolle's Theorem)**.** *Suppose that f is*

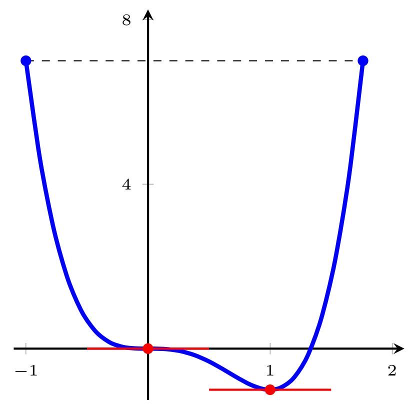
1. *continuous on the closed interval $[a, b]$,*
2. *differentiable on the open interval (a, b),*
3. *and satisfies $f(a) = f(b)$.*

Then there is a number c in (a, b) so that

$$f'(c) = 0.$$

Intuitively, this says that a function that starts and ends at the same value must have a zero derivative at some point in that interval.





Example 4.2.2. A particle moving along a straight line starts in motion. At time t, the particle is at the same starting position. By Rolle's Theorem, there must be a time in $(0, t)$ at which the particle's velocity was 0.

This theorem can be generalized by imagining what would happen if we took the graph of our function f on $[a, b]$ and rotated it. The horizontal tangent line would rotate as well and would still be parallel to the secant line between $(a, f(a))$ and $(b, f(b))$. The following theorem states this quite concretely.

Theorem 4.2.3 (Mean Value Theorem)**.** *Let f be a function that satisfies the following hypotheses:*

1. *f is continuous on $[a, b]$.*
2. *f is differentiable on (a, b).*

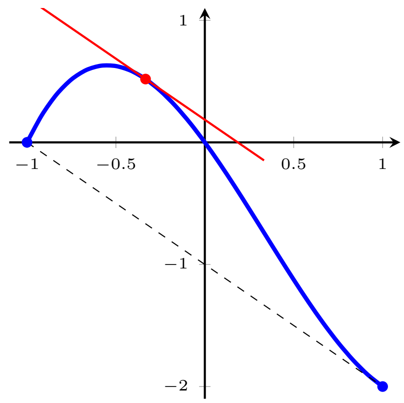
Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The number on the right-hand side of the equals sign is the slope of the line between $(a, f(a))$ and $(b, f(b))$. This slope is sometimes **average slope** or **mean slope**, which is where the theorem gets its name.

Remark. In the case where $f(a) = f(b)$, the Mean Value Theorem is exactly Rolle's Theorem

Example 4.2.4. Consider the function $f(x) = x^3 - x^2 - 2x$ on $[-1, 1]$. Find the value(s) of c that satisfies the conclusion of the Mean Value Theorem (MVT).



We first note that since $f(x)$ is a polynomial, it is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$, so we can apply the Mean Value Theorem. Now, we have

$$f'(x) = 3x^2 - 2x - 2,$$

and

$$\frac{f(1) - f(-1)}{1 - (-1)} = \frac{-2 - 0}{2} = -1.$$

By the Mean Value Theorem there exists some number c in $(-1, 1)$ so that

$$\begin{aligned} 3c^2 - 2c - 2 &= -1 \\ 3c^2 - 2c - 1 &= 0 \\ (3c + 1)(c - 1) &= 0, \end{aligned}$$

so $c = -\frac{1}{3}$. ($c \neq 1$ since the Mean Value Theorem only gives us c-values in $(-1, 1)$.)

Example 4.2.5. Does there exist a continuous function f such that $f(1) = 6$, $f(7) = 9$, and $f'(x) \geq 2$ for all x in $(1, 7)$?

No. Since f satisfies the conditions of the mean value theorem, there must exist c in the interval $(1, 7)$ such that

$$f'(c) = \frac{f(7) - f(1)}{7 - 1} = \frac{9 - 6}{6} = \frac{3}{6} = \frac{1}{2}.$$

As such, it is impossible for $f'(x) \geq 2$ for all c in $(1, 7)$.

4.3 Derivatives and the Shapes of Graphs

4.3.1 What Does f' Say About f?

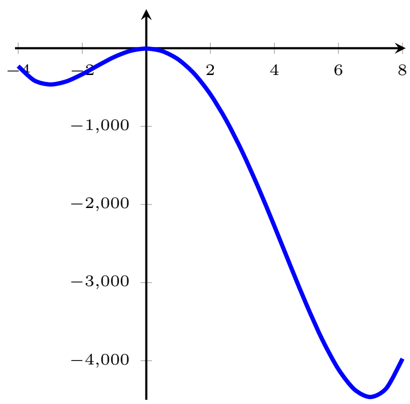
Notice that, for a given graph of a function, the slope of the tangent line is positive when the function values are increasing, and the slope of the tangent line is negative when the function values are decreasing. Derivatives allow us to figure out where these intervals of increase/decrease occur.

Proposition 4.3.1 (Increasing/Decreasing Test)**.**

a. *If $f'(x) > 0$ on an interval (a, b), then f is increasing on the interval (a, b).*

b. *If $f'(x) < 0$ on an interval (a, b), then f is decreasing on the interval (a, b).*

Example 4.3.2. Find the intervals of increase and decrease for the function $f(x) = 3x^4 - 16x^3 - 126x^2 - 5$.



First, we take the derivative.

$$f'(x) = 12x^3 - 48x^2 - 252x = 12x(x + 3)(x - 7)$$

By the Increasing/Decreasing test, it suffices to find intervals for which $f'(x) < 0$ and $f'(x) > 0$. This means that the intervals to test occur between critical points. Since $f'(x) = 0$ for $x = -3, 0, 7$, we consider the four intervals $(-\infty, -3)$, $(-3, 0)$, $(0, 7)$, and $(7, \infty)$.

By the Intermediate Value Theorem, it suffices to check a single value in each interval to determine the derivative's sign on each interval. We'll check -4, -1, 1, and 8.

$$\begin{aligned} f'(-4) &= -528 \\ f'(-1) &= 192 \\ f'(1) &= -288 \\ f'(8) &= 1056 \end{aligned}$$

Interval	$f'(x)$	f
$(-\infty, -3)$	$-$	decreasing
$(-3, 0)$	$+$	increasing
$(0, 7)$	$-$	decreasing
$(7, \infty)$	$+$	increasing

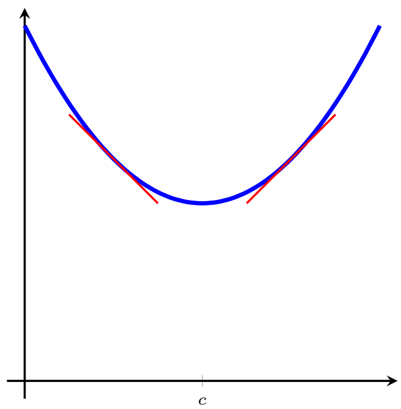
What we notice from this last example is that our local maxima/minima occur between sign changes for our derivative. This leads us to the following test for local extrema.

Theorem 4.3.3 (First Derivative Test)**.** *Suppose that f is continuous and that c is a critical point of f.*

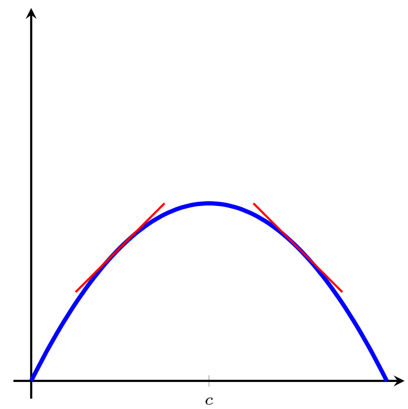
a. *If f' changes from positive to negative at c, then f has a local maximum at c.*

b. *If f' changes from negative to a positive at c, then f has a local minimum at c.*

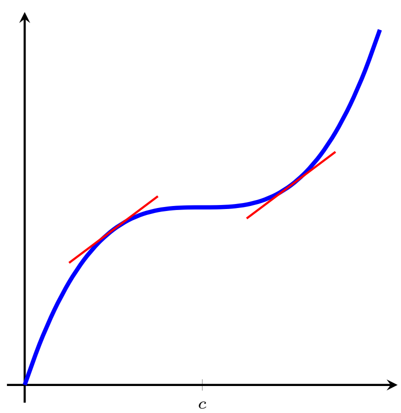
c. *If f' does not change sign at c, then f has no local minimum or maximum at c.*



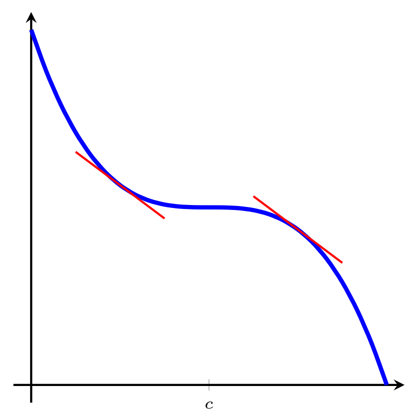
Local minimum



Local maximum



No local maximum/minimum



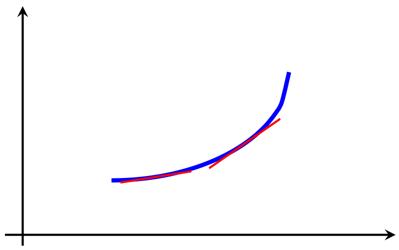
No local maximum/minimum

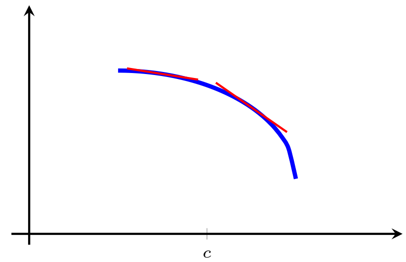
Example 4.3.4. Use the First Derivative Test to find minimum and maximum values of the function f in Example 4.3.2.

From the chart, we see that $f'(x)$ changes from negative to positive at -3, so -3 is a local minimum with value $f(-3) = -464$. Also, $f'(x)$ changes from positive to negative at 0, so 0 is a local maximum with value $f(0) = -5$. Finally, $f'(x)$ changes from negative to positive at 7, so 7 is a local minimum with value $f(7) = -4464$.

4.3.2 What Does f'' Say About f'?

In the two graphs below, both functions have a positive first derivative, and yet there's something fundamentally different about them.





Definition. If the graph of f lies above all of its tangents on an interval, I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

Since curves can have multiple intervals on which they are concave upward or concave downward, we give a special name to these points where the concavity changes.

Definition. A point on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward (or *vice-versa*) at that point.

Using the second derivative, we get the following test for concavity.

Proposition 4.3.5 (Concavity Test)**.**

a. *If $f''(x) > 0$ for all x in an interval I, then the graph of f is concave upward on I.*

b. *If $f''(x) > 0$ for all x in an interval I, then the graph of f is concave upward on I.*

Example 4.3.6. Find the intervals of concavity for function f in Example 4.3.2. First, we take the second derivative.

$$f''(x) = 36x^2 - 96x - 252$$

By the Concavity test, it suffices to find intervals for which $f''(x) < 0$ and $f''(x) > 0$. This means that the intervals to test occurs between x-values for which $f''(x) = 0$. The Quadratic Formula gives us that $f''(x)$ has roots at $x = \frac{4}{3} - \frac{\sqrt{79}}{3} \approx -1.629$ and at $x = \frac{4}{3} + \frac{\sqrt{79}}{3} \approx 4.296$. Thus the intervals to consider are $(-\infty, -1.629)$, $(-1.629, 4.296)$, and $(4.296, \infty)$.

By the Intermediate Value Theorem, it suffices to check a single value in each interval to determine the second derivative's sign on each interval. We'll check -2, 0, and 5.

$$f''(-2) = 84$$
$$f''(0) = -252$$
$$f''(5) = 168$$

Interval	$f'(x)$	f
$\left(-\infty, \frac{4}{3} - \frac{\sqrt{79}}{3}\right)$	$+$	concave up
$\left(\frac{4}{3} - \frac{\sqrt{79}}{3}, \frac{4}{3} + \frac{\sqrt{79}}{3}\right)$	$-$	concave down
$\left(\frac{4}{3} + \frac{\sqrt{79}}{3}, \infty\right)$	$+$	concave up

Remark. By the Intermediate Value Theorem, we can deduce that inflection points must occur where $f''(x) = 0$. Like local minima/maxima, this condition alone is not enough to test whether points are inflection points.

Going back to maxima/minima, the second derivative and concavity give us the following useful test.

Theorem 4.3.7 (Second Derivative Test)**.** *Suppose f'' is continuous near c.*

a. *If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.*

b. *If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.*

Remark. The second derivative test is inconclusive when $f''(c) = 0$ or $f''(c)$ does not exist. There may be a local maximum, local minimum, or neither at c. In these instances, we may have to resort to using the First Derivative Test

Example 4.3.8. Use the second derivative test to find relative extrema for function f in Example 4.3.2.

We know that $f'(c) = 0$ for $c = -3, 0, 7$. Since $f''(-3) = 360 > 0$, we have that f has a local minimum at 3 by the Second Derivative test. Similarly, $f''(0) = -252 < 0$, so we have that f has a local maximum at 0. Lastly, $f''(7) = 840 > 0$, so we have that f has a local minimum at 7. This agrees with the results from the First Derivative Test as in Example 4.3.4.

4.4 Curve Sketching

Putting it all together, we can now use this information to get a feel for the shape of the graph.

Example 4.4.1. Sketch a graph of $f(x) = \dfrac{x^2 - 2x + 4}{x - 2}$.

We first need to collect a bit of information.

a. The domain of f is $(-\infty, 2) \cup (2, \infty)$.

b. The numerator is never 0, so f does not have any x-intercepts. Since $f(0) = -2$, f has a y-intercept of -2.

c. Since f is a rational function and the degree of the numerator is greater than the degree of the denominator, f has no horizontal asymptotes.

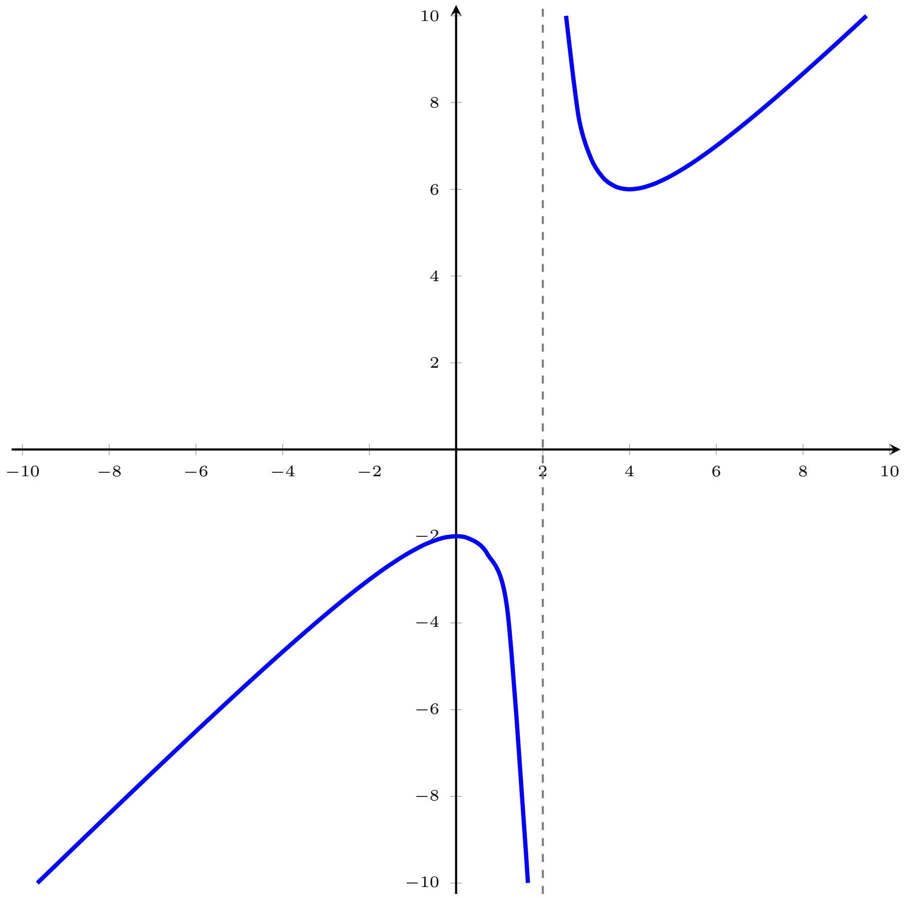
d. Since $\lim_{x \to 2^-} f(x) = -\infty$ and $\lim_{x \to 2^+} f(x) = \infty$, f has a vertical asymptote $x = 2$.

e. With the quotient rule, we see that $f'(x) = \dfrac{x(x-4)}{(x-2)^2}$, so we have critical points at $x = 0$ and 4. Combining this with our domain and testing a number in each interval, we see that f must be increasing on $(-\infty, 0)$, decreasing on $(0, 2)$, decreasing on $(2, 4)$ and increasing on $(4, \infty)$.

f. By the first derivative test, 0 corresponds to a local maximum with value -2, and 4 corresponds to a local minimum with value 6.

g. Again with the quotient rule, we get that $f''(x) = \dfrac{8}{(x-2)^3}$. Since the numerator is never zero, $f''(x) \neq 0$, and so we can have no inflection points. Combining this information with our domain and testing a number in each interval, we see that f must be concave downward on $(-\infty, 2)$ and concave upward on $(2, \infty)$.

Interval	**Characteristic of Graph**
$(-\infty, 2)$	Increasing, Concave Down
$x = 0$	Local Maximum (value of -2)
$(0, 2)$	Decreasing, Concave Down
$x = 2$	Vertical Asymptote
$(2, 4)$	Decreasing, Concave Up
$x = 4$	Local Minimum (value of 6)
$(4, \infty)$	Increasing, Concave Up


10
8
6
4
2
−10
−8
−6
−4
−2
2
4
6
8
10
−2
−4
−6
−8
−10

4.5 Optimization Problems

Another important application of derivatives is in optimization problems. When doing these types of problems, it's important to keep in mind the two pieces of information that you'll need - the function to be optimized, and the constraints.

Example 4.5.1. Find two positive numbers whose product is 169 and whose sum is a minimum.

First we'll call these numbers x and y. We're looking to find the minimum of $x + y$, subject to the constraints that $xy = 169$ and x, y in $(0, \infty)$. Rearranging our constraint, we see that $y = \frac{169}{x}$, so we can rewrite our sum as the function

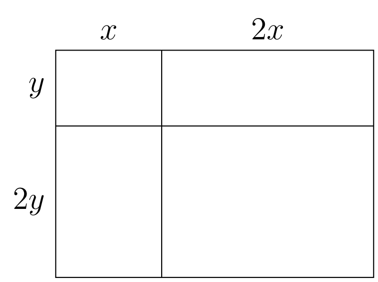
$$f(x) = x + y = x + \frac{169}{x}.$$

To find the minimum on the interval $(0, \infty)$, we take the derivative and set it equal to 0.

$$\begin{aligned} f'(x) = 1 - \frac{169}{x^2} &= 0 \\ 1 &= \frac{169}{x^2} \\ x^2 &= 169 \\ \Rightarrow \quad x &= 13. \end{aligned}$$

Plugging this back into our constraint equation, $y = \frac{169}{x} = \frac{169}{13} = 13$. So, the pair of numbers whose product is 169 and whose sum is a minimum is $x = 13$, $y = 13$.

Example 4.5.2. A farmer has 360 feet of fencing with which to build the pen shown below. What is the maximum area that the farmer can enclose?



We're trying to optimize the area $A = (3x)(3y)$, subject to the constraints that $3(3x) + 3(3y) = 360$ and x, y in $[0, 40]$. Rearranging our constraint, we see that $y = 40 - x$, so we can rewrite our area as the function

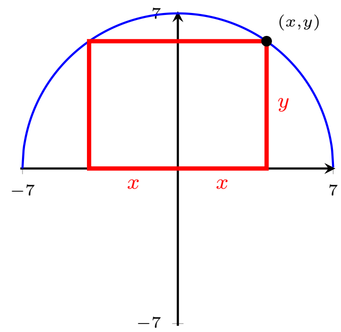
$$A(x) = (3x)(3y) = (3x)(120 - 3x) = 360x - 9x^2.$$

To find the maximum on the interval $[0, 40]$, we start by taking the derivative and setting it equal to 0.

$$\begin{aligned} A'(x) = 360 - 18x &= 0 \\ 360 &= 18x \\ \Rightarrow \quad x &= 20. \end{aligned}$$

By the Extreme Value Theorem, the maximum enclosed area must occur when $x = 0$, $x = 20$, or $x = 40$. Indeed, by comparing these values, we have that the maximum occurs at $x = 20$, and the maximum area enclosed is $3600\,\text{ft}^2$.

Example 4.5.3. A rectangle is bounded between the x-axis and the semicircle $y = \sqrt{49 - x^2}$. What length and width should the rectangle have so that its area is maximum?



We're trying to maximize the area $A = (2x)(y)$ subject to the constraints that x, y in $[0, 7]$ and $y = \sqrt{49 - x^2}$. With this constraint in mind, our area function becomes

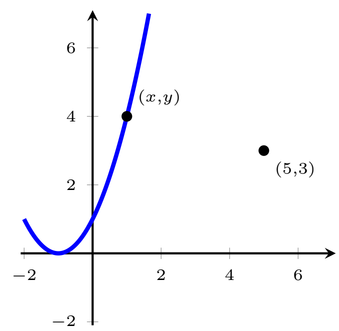
$$A(x) = (2x)(y) = 2x\sqrt{49 - x^2}.$$

Taking the derivative and setting it equal to 0, we see that

$$A'(x) = 2\sqrt{49 - x^2} + \frac{-2x^2}{\sqrt{49 - x^2}} = 02\sqrt{49 - x^2} \qquad = \frac{2x^2}{\sqrt{49 - x^2}}$$
$$2(49 - x^2) = 2x^2$$
$$49 - x^2 = x^2$$
$$2x^2 = 49$$
$$\Rightarrow \quad x = \frac{7\sqrt{2}}{2} \approx 4.9497.$$

This means that the maximum area occurs when $x = 0$, $x = \frac{7\sqrt{2}}{2}$, or $x = 7$. By plugging each of these three values into $A(x)$, we see that it occurs when $x = \frac{7\sqrt{2}}{2}$. So the dimensions of the rectangle are $7\sqrt{2}$ by $\frac{7\sqrt{2}}{2}$, or approximately 9.8995 by 4.9497.

Example 4.5.4. Find the point on the graph of $y = (x+1)^2$ that is closest to the point $(5, 3)$.



We're minimizing the distance, z, between a point (x, y) on $y = (x+1)^2$ and $(5, 3)$. Recall that $z = \sqrt{(x-5)^2 + (y-3)^2}$. By plugging $y = (x+1)^2$, we get

$$z(x) = \sqrt{(x-5)^2 + ((x+1)^2 - 3)^2} = \sqrt{(x-5)^2 + (x^2 + 2x - 2)^2},$$

whence

$$z'(x) = \frac{2(x-5) + 2(x^2+2x-2)(2x+2)}{\sqrt{(x-6)^2 + (x^3-3)^2}} = \frac{2(x-1)(2x^2+8x+9)}{\sqrt{(x-6)^2 + (x^3-3)^2}}.$$

We see that $z'(x) = 0$ has only one positive real solution, and it is when $x = 1$. By the first derivative test, we see that $x = 1$ does indeed correspond to a local minimum, so it follows that the point $(1, 4)$ is the closes point on the graph of $y = (x+1)^2$ to $(5, 3)$.

4.7 AntiDerivatives

In a totally-not-contrived scenario, suppose your friend throws a ball straight into the air, tells you the function of the ball's velocity $v(t)$, and asks you about the position of the ball at any given time. Since we know that velocity is the first derivative of the position function, the question boils down to finding some function $p(t)$ so that $v(t) = p'(t)$.

Definition. A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Example 4.7.1. Find an antiderivative F for the function $f(x) = x^3$.

Just keeping in mind the power rule, we can see that $F(x) = \frac{1}{4}x^4$ is an antiderivative for f.

Of course, because the derivative of a constant is 0, $F(x) = \frac{1}{4}x^4 + 100$ is also an antiderivative for f. This tells us that antiderivatives are not unique and can differ by a constant. In the totally-not-contrived scenario, the interpretation is this: you can tell what the position of the ball is exactly if you know what the initial height of your friend's throw was, and that could change depending on whether s/he was standing on top of a building, at sea level, or at the bottom of the Bingham Canyon Mine.

Proposition 4.7.2. *If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is*

$$F(x) + C,$$

where C is an arbitrary constant.

Example 4.7.3. Find the most general antiderivative of each of the following functions.

a. $f(x) = x^n$, $n \neq -1$

b. $f(x) = \frac{1}{x}$

c. $f(x) = \sin x$

a. From the power rule, we know that $F(x) = Ax^{n+1}$ should be an antiderivative. In particular, $F'(x) = A(n+1)x^n = x^n$, so $A = \frac{1}{n+1}$. By Proposition 4.7.2, the most general antiderivative of f is

$$F(x) = \frac{1}{n+1}x^{n+1}.$$

b. The power rule doesn't apply here (if it did, it would say that $\frac{d}{dx}[\frac{1}{1}x^0] = \frac{1}{x}$, but x^0 is a constant). Instead, we recall that the function with derivative $\frac{1}{x}$ is $\ln|x|$. So, by Proposition 4.7.2, the most general antiderivative of f is

$$F(x) = \ln|x| + C.$$

c. Since sine and cosine alternate with their derivatives, we know that $F(x) = A\cos x$ for some constant A should be the antiderivative. Since $F'(x) = -A\sin x = \sin x$, we get that $A = -1$. So, by Proposition 4.7.2, the most general antiderivative of f is

$$F(x) = -\cos x + C.$$

Playing this same game, we can complete the following useful table of antiderivatives:

Function	Particular Antiderivative	Function	Particular Antiderivative
$cf(x)$	$cF(x)$	$\sin x$	$-\cos x$
$f(x)+g(x)$	$F(x)+G(x)$	$\cos x$	$\sin x$
x^n, $(n \neq 1)$	$\frac{1}{n+1}x^{n+1}$	$\sec^2 x$	$\tan x$
$\frac{1}{x}$	$\ln \lvert x \rvert$	$\sec x \tan x$	$\sec x$
e^x	e^x	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
		$\frac{1}{1+x^2}$	$\tan^{-1} x$

Example 4.7.4. Find all functions g such that

$$g'(x) = 4\cos x + \frac{20x^4 - \sqrt{x^3}}{x}.$$

First we'll perform some algebraic manipulation to g' to get

$$g'(x) = 4\cos x + \frac{20x^4}{x} - \frac{\sqrt{x^3}}{x} = 4\cos x + 20x^3 - x^{1/2}.$$

Using the table rules above, we see that

$$g(x) = 4\sin x + 20 \cdot \frac{1}{4}x^4 - \frac{1}{3/2}x^{3/2} + C = 4\sin x + 5x^4 - \frac{2}{3}x^{3/2} + C$$

is our most general form of the antiderivative and thus gives us all functions g.

Example 4.7.5. An object, initially at rest, falls off of a 200 foot building and constantly accelerates at $-32\,\text{ft/s}^2$. Find the equation of the position function.

We're given that the acceleration function for the object is

$$a(t) = -32.$$

This is the first derivative of the velocity function, so the antiderivative of $a(t)$ gets us

$$v(t) = -32t + C,$$

for some constant C. Since the object is initially at rest, $v(0) = 0$, so $C = 0$. Now, this function is the first derivative of the position function, so the antiderivative of $v(t)$ gets us

$$p(t) = -16t^2 + D,$$

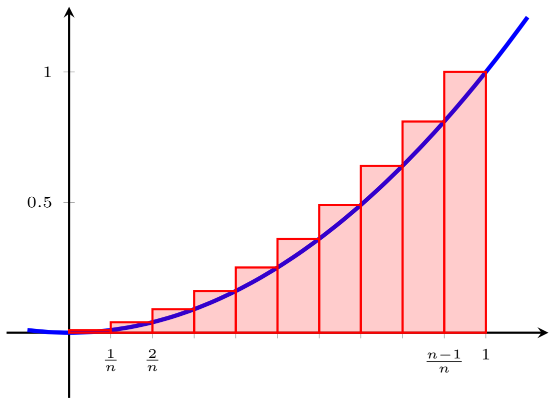
for some constant D. Since $p(0) = 200$, we deduce that $D = 200$. Thus, the precise position function for the object is $p(t) = -16t^2 + 200$, which exactly agrees with what we know about the kinematic motion equation from previous examples.

5 Integrals

5.1 Area and Distances

We consider the graph of $y = x^2$ over the interval $[0, 1]$.

Suppose we partition $[0, 1]$ into n-many smaller intervals and let R_n be the sum of the areas of rectangles as shown in the picture.



Example 5.1.1. Show that $\lim_{n\to\infty} R_n = \frac{1}{3}$.

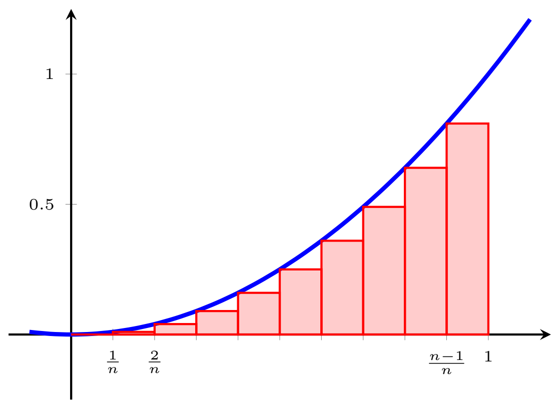
We see that

$$\begin{aligned} R_n &= \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2 \\ &= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right) \\ &= \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} && \text{(sum of the first } n \text{ squares)} \\ &= \frac{(n+1)(2n+1)}{6n^2} \\ &= \frac{2n^2 + 3n + 1}{6n^2}. \end{aligned}$$

We then have that

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{2n^2 + 3n + 1}{6n^2} = \frac{2}{6} = \frac{1}{3}.$$

Still partitioning $[0, 1]$ into n-many smaller intervals and let L_n be the sum of the areas of rectangles as shown in the picture.



Example 5.1.2. Show that $\lim_{n\to\infty} L_n = \dfrac{1}{3}$.

We see that

$$\begin{aligned} R_n &= \frac{1}{n}\left(\frac{0}{n}\right)^2 + \frac{1}{n}\left(\frac{1}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n-1}{n}\right)^2 \\ &= \frac{1}{n^3}\left(1^2 + \cdots + (n-1)^2\right) \\ &= \frac{1}{n^3} \cdot \frac{(n-1)((n-1)+1)(2(n-1)+1)}{6} && \text{(sum of the first } n-1 \text{ squares)} \\ &= \frac{(n-1)(2n-1)}{6n^2} \\ &= \frac{2n^2-3n+1}{6n^2}. \end{aligned}$$

We then have that

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} \frac{2n^2-3n+1}{6n^2} = \frac{2}{6} = \frac{1}{3}.$$

It appears that as n increases, L_n and R_n both become better and better approximations of the area under the curve. So, we define the area A to be the limit of the sums of the approximating rectangles. In other words,

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n.$$

Thus, the region in the previous two examples has area $\frac{1}{3}$.

5.1.1 Area in General

Suppose we have a curve $y = f(x)$ on an interval $[a, b]$. Dividing it into n-many smaller intervals, we have that each interval has width

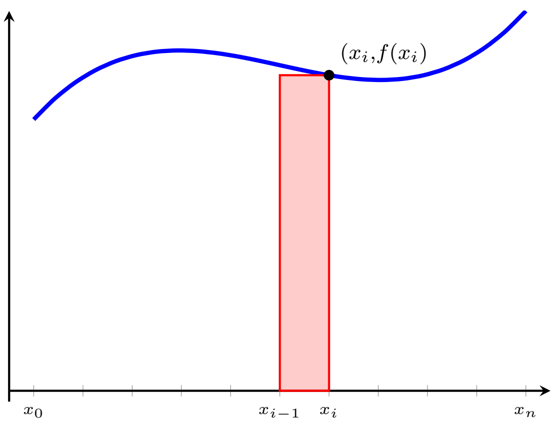
$$\Delta x = \frac{b-a}{n}.$$

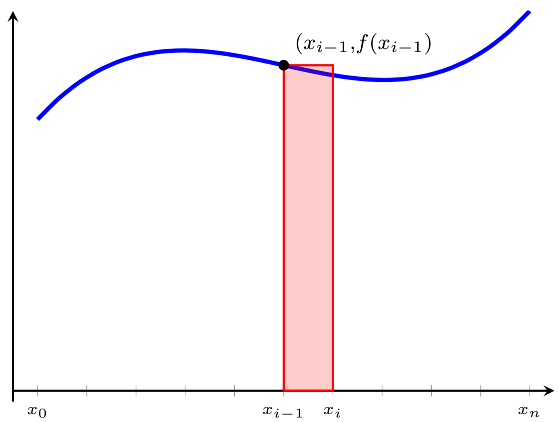
This means that we can divide $[a, b]$ into intervals

$$[x_0, x_1],\ [x_1, x_2],\ \ldots,\ [x_{n-1}, x_n]$$

where $a = x_0$, $b = x_n$, and $x_i = a + i\Delta x$ for $i = 0, \ldots, n$. We then define R_n and L_n as follows:

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \qquad L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x$$





Definition. The **area** A of the region that lies under the graph of the continuous function f is the limit of the sum of the areas of the approximating rectangles

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n.$$

Remark. In our definition, we focus on obtaining the area using only the left or only the right endpoint of each interval. In fact, it actually doesn't matter which point you choose in the interval - even arbitrary points in the interior will result in the same limit.

To avoid writing out all these terms, we use **sigma notation** to keep these things more compact.

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

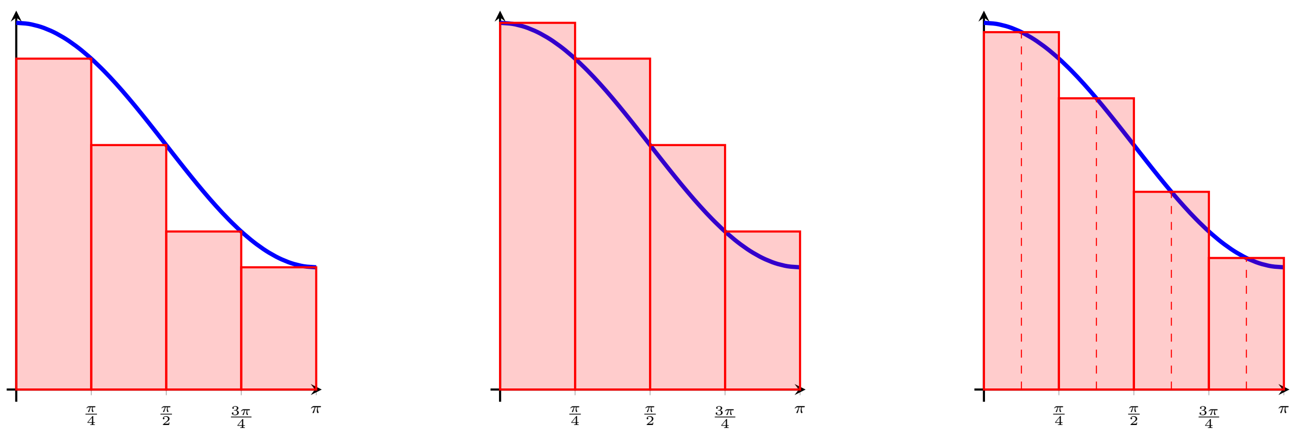
So, this means that we can rephrase the limit for area slightly as:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x.$$

Example 5.1.3. Estimate the area under the graph of $f(x) = 2 + \cos x$ from $x = 0$ to $x = \pi$ using 4 rectangles and

a. right endpoints. **b.** left endpoints. **c.** midpoints.



We note that $\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}$, so $x_0 = 0$, $x_1 = \frac{\pi}{4}$, $x_2 = \frac{\pi}{2}$, $x_3 = \frac{3\pi}{4}$, and $x_4 = \pi$.

1. For the right endpoints, we have

$$
\begin{aligned}
R_4 = \sum_{i=1}^{4} f(x_i)\Delta x &= \sum_{i=1}^{4} \left(2 + \cos\left(\frac{i\pi}{4}\right)\right) \cdot \frac{\pi}{4} \\
&= \frac{\pi}{4}\left(2 + \cos\frac{\pi}{4} + 2 + \cos\frac{\pi}{2} + 2 + \cos\frac{3\pi}{4} + 2 + \cos\pi\right) \\
&= \frac{7\pi}{4} \approx 5.4978.
\end{aligned}
$$

2. For the left endpoints, we have

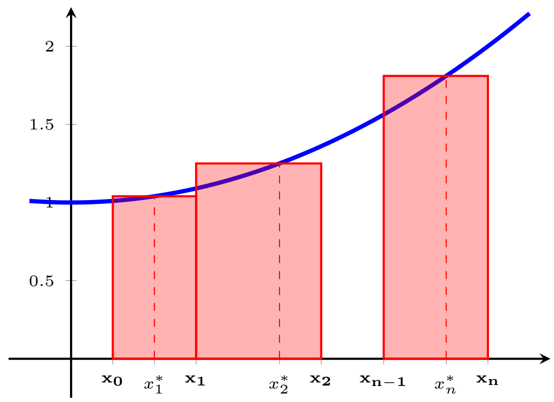
$$
\begin{aligned}
L_4 = \sum_{i=1}^{4} f(x_{i-1})\Delta x &= \sum_{i=1}^{4} \left(2 + \cos\left(\frac{(i-1)\pi}{4}\right)\right) \cdot \frac{\pi}{4} \\
&= \frac{\pi}{4}\left(2 + \cos 0 + 2 + \cos\frac{\pi}{4} + 2 + \cos\frac{\pi}{2} + 2 + \cos\frac{3\pi}{4}\right) \\
&= \frac{9\pi}{4} \approx 7.0686.
\end{aligned}
$$

3. For the midpoints, we have

$$
\begin{aligned}
M_4 = \sum_{i=1}^{4} f\left(\frac{x_i - x_{i-1}}{2}\right)\Delta x &= \sum_{i=1}^{4} \left(2 + \cos\left(\frac{(2i-1)\pi}{8}\right)\right) \cdot \frac{\pi}{4} \\
&= \frac{\pi}{4}\left(2 + \cos\frac{\pi}{8} + 2 + \cos\frac{3\pi}{8} + 2 + \cos\frac{5\pi}{8} + 2 + \cos\frac{7\pi}{8}\right) \\
&= 2\pi \approx 6.2832.
\end{aligned}
$$

5.2 The Definite Integral

As we talked about before, we chose right and left endpoints or midpoints for approximating the area under the curve, but it ultimately didn't even matter - we could base our rectangles at a height chosen by any number x_i^* chosen in the evenly-spaced interval $[x_{i-1}, x_i]$. In fact, as it turns out, we don't even have to evenly space our intervals, as shown in the picture below. So, instead of each interval having length Δx, we let $\Delta x_i = x_i - x_{-1}$ be the length of the i^{th} interval.



Definition. If f is defined on $[a, b]$, then the **definite integral of f from a to b** is the number

$$\int_a^b f(x)\,dx = \lim_{\max \Delta_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

If the limit exists, we say that f is **integrable on** $[\mathbf{a}, \mathbf{b}]$. Here $\int$ is called the **integral sign**, f is called the **integrand**, and a and b are called the **limits of integration** (in particular, a is called the **lower limit** and b is called the **upper limit**). For now, dx has no meaning by itself.

Indeed, computationally, it's often much simpler to suppose that we pick our x_i's so that Δx_i is the same for all i, and for x_i^* to always be one of the endpoints or the midpoint of each interval. If x_i^* is the right (respectively, left) endpoint, we call this sum a **right** (respectively, **left**) **Riemann Sum**.

Proposition 5.2.1. *If f is integrable on $[a, b]$, then we can choose $\Delta x = \frac{b-a}{n}$ so that*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Remark. With this assumption, we'll usually just assume the right endpoint and write $x_i^* = x_i$.

Proposition 5.2.2. *If f is continuous or has only finitely many jump discontinuities on $[a, b]$, then f is integrable on $[a, b]$, that is $\int_a^b f(x)\,dx$ exists.*

This is great, because it says that it's easier to be integrable than it is to be differentiable. Moreover, many functions we've worked with in this class satisfies these properties.

Example 5.2.3. Express

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{3}{x_i}\right) \Delta x$$

as a definite integral on the interval $[1, 5]$

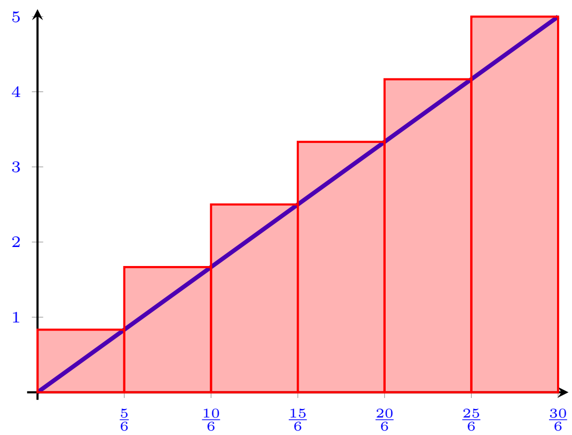
$$\int_1^5 \left(1 + \frac{3}{x}\right) dx$$

Example 5.2.4. Approximate the definite integral using a right Riemann sum with 6 rectangles. Then do it again with n rectangles to and evaluate the definite integral exactly.

$$\int_0^5 x\, dx$$

We have here that $f(x) = x$, $\Delta x = \frac{5-0}{6} = \frac{5}{6}$, and

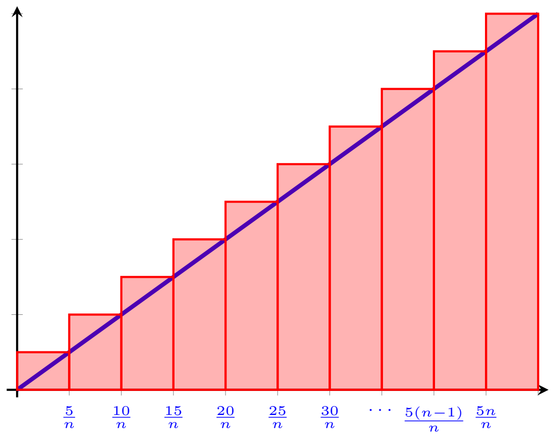
$$x_0 = 0, \quad x_1 = \frac{5}{6}, \quad x_2 = \frac{10}{6}, \quad x_3 = \frac{15}{6}, \quad x_4 = \frac{20}{6}, \quad x_5 = \frac{25}{6}, \quad x_n = \frac{30}{6} = 5.$$



So, the sum of the areas of these rectangles is

$$\begin{aligned}
\sum_{i=1}^{6} f(x_i)\,\Delta x &= f(x_1)\,\Delta x + f(x_2)\,\Delta x + f(x_3)\,\Delta x + f(x_4)\,\Delta x + f(x_5)\,\Delta x + f(x_6)\,\Delta x \\
&= f\left(\frac{5}{6}\right)\frac{5}{6} + f\left(\frac{10}{6}\right)\frac{5}{6} + f\left(\frac{15}{6}\right)\frac{5}{6} + f\left(\frac{20}{6}\right)\frac{5}{6} + f\left(\frac{25}{6}\right)\frac{5}{6} + f\left(\frac{30}{6}\right)\frac{5}{6} \\
&= \left(\frac{5}{6}\right)\frac{5}{6} + \left(\frac{10}{6}\right)\frac{5}{6} + \left(\frac{15}{6}\right)\frac{5}{6} + \left(\frac{20}{6}\right)\frac{5}{6} + \left(\frac{25}{6}\right)\frac{5}{6} + \left(\frac{30}{6}\right)\frac{5}{6} \\
&= \frac{175}{12} \approx 15.5833.
\end{aligned}$$

Now, to do this with n rectangles, we do the same thing as before. Here $f(x) = x$, $\Delta x = \frac{5-0}{n} = \frac{5}{n}$, $x_0 = 0$, and $x_i = x_0 + i \cdot \Delta x = \frac{5i}{n}$.



So, the sum of the areas of these rectangles is

$$\begin{aligned}
\int_0^5 x\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \left[\left(\frac{5}{n}\right)\frac{5}{n} + \left(\frac{10}{n}\right)\frac{5}{n} + \cdots + \left(\frac{5(n-1)}{n}\right)\frac{5}{n} + \left(\frac{5n}{n}\right)\frac{5}{n} \right] \\
&= \lim_{n\to\infty} \left(\frac{25}{n^2}\right)[1 + 2 + \cdots + (n-1) + n] \\
&= \lim_{n\to\infty} \left(\frac{25}{n^2}\right)\left(\frac{n^2+n}{2}\right) && \text{(sum of first } n \text{ positive integers)} \\
&= \lim_{n\to\infty} \frac{25n^2 + 25n}{2n^2} = \frac{25}{2} = 12.5.
\end{aligned}$$

And indeed, $\frac{25}{2}$ is the area we got by just knowing about the area of a triangle.

Proposition 5.2.5 (Properties of Definite Integrals)**.** *Let $f(x)$, $g(x)$ be integrable functions on $[a, b]$ and c a real number. Then*

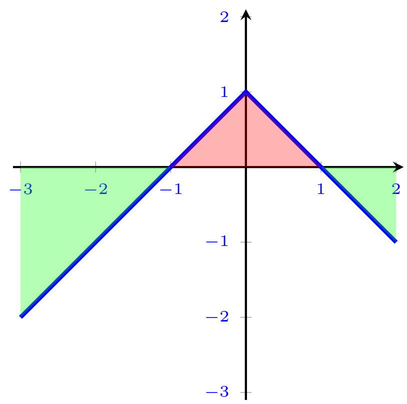
1. $\int_a^b c\,dx = c(b-a)$
2. $\int_a^b c \cdot f(x)\,dx = c\int_a^b f(x)\,dx$
3. $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
4. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

Proof. The proof of each property follows easily from properties of summations and the algebra of limits. Only the last one seems kind of strange, but indeed it comes from the fact that when looking at $\int_a^b$, we have $\Delta x = \frac{b-a}{n}$ and when looking at $\int_b^a$, we have $\Delta x = \frac{a-b}{n} = -\frac{b-a}{n}$. □

When $f(x)$ is positive on $[a, b]$, the definite integral $\int_a^b f(x)\,dx$ represents the area above the x-axis and under the curve $y = f(x)$. When $f(x)$ takes both positive and negative values on $[a, b]$, the definite integral $\int_a^b f(x)\,dx$ represents the net area (that is, the area above x-axis and below the curve $y = f(x)$, minus the area below the x-axis and above the curve $y = f(x)$).

Example 5.2.6. Evaluate the following integrals by interpreting it in terms of area: $\displaystyle\int_{-3}^{2}(1 - |x|)\,dx$

Drawing this out, we see that we just have the areas of red triangle minus the area of the green triangles.



So,

$$\int_{-3}^{2}(1 - |x|)\,dx = -\frac{1}{2}(2)(2) + \frac{1}{2}(2)(1) - \frac{1}{2}(1)(1) = -\frac{3}{2}.$$

This next result tells us how we can combine integrals on adjacent intervals.

Proposition 5.2.7. *Suppose $f(x)$ is integrable on $[a, b]$ and c is in $[a, b]$. Then certainly f is integrable on both $[a, c]$ and $[c, b]$ and*

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$$

Example 5.2.8. Suppose $\int_0^{27} f(x)\,dx = 10$ and $\int_0^{15} f(x)\,dx = 3$. Find $\int_{15}^{27} f(x)\,dx$.

By Proposition 5.2.7,

$$\int_0^{15} f(x)\,dx + \int_{15}^{27} f(x)\,dx = \int_0^{27} f(x)\,dx$$
$$3 + \int_{15}^{27} f(x)\,dx = 10$$
$$\int_{15}^{27} f(x)\,dx = 7$$

Example 5.2.9. Suppose $\int_0^4 f(x)\,dx = 2$, $\int_4^6 f(x)\,dx = 3$, and $\int_0^6 g(x)\,dx = 9$, find

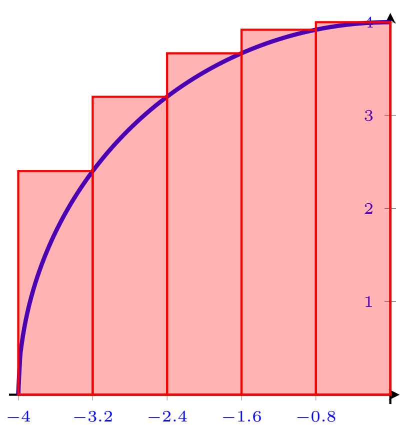
$$\int_0^6 3f(x) - g(x)\,dx.$$

$$\begin{aligned}
\int_0^6 (3f(x) - g(x) + 1)\ dx &= \int_0^6 3f(x)\,dx - \int_0^6 g(x)\,dx + \int_0^6 1\,dx \\
&= 3\int_0^6 f(x)\,dx - \int_0^6 g(x)\,dx + \int_0^6 1\,dx \\
&= 3\left(\int_0^4 f(x)\,dx + \int_4^6 f(x)\,dx\right) - \int_0^6 g(x)\,dx + \int_0^6 1\,dx \\
&= 3(2+3) - 9 + 6 = 12.
\end{aligned}$$

Example 5.2.10. Approximate the following definite integral using a right Riemann sum with 5 rectangles: $\displaystyle\int_{-4}^{0} \sqrt{16 - x^2}\,dx$.

We have here that $f(x) = \sqrt{16 - x^2}$, $\Delta x = \frac{0-(-4)}{5} = 0.8$, and

$$x_0 = -4, \quad x_1 = -3.2, \quad x_2 = -2.4, \quad x_3 = -1.6, \quad x_4 = -0.8, \quad x_5 = 0.$$

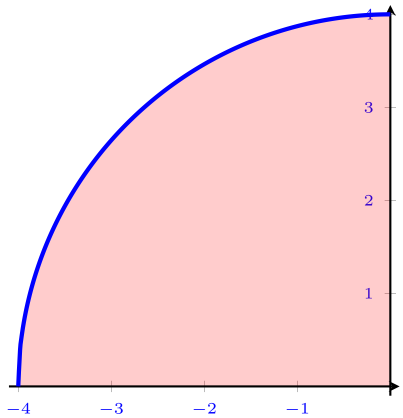


So, the sum of the areas of these rectangles is

$$\begin{aligned}
\sum_{i=1}^{5} f(x_i)\,\Delta x &= f(x_1)\,\Delta x + f(x_2)\,\Delta x + f(x_3)\,\Delta x + f(x_4)\,\Delta x + f(x_5)\,\Delta x \\
&= f(-4)\,0.8 + f(-3.2)\,0.8 + f(-2.4)\,0.8 + f(-1.6)\,0.8 + f(-0.8)\,0.8 + f(0)\,0.8 \\
&\approx (2.4)\,0.8 + (3.2)\,0.8 + (3.3667)\,0.8 + (3.919)\,0.8 + (4)\,0.8 + \\
&\approx 13.748
\end{aligned}$$

Example 5.2.11. Evaluate the definite integral in Example 5.2.10 by interpreting it in terms of area.

Notice that, if $y = \sqrt{16 - x^2}$, then $y^2 = 16 - x^2$, and thus $x^2 + y^2 = 16$, so the graph traces out the top half of the circle of radius 4. Drawing this out, we see that we're just looking at the area under one quarter of this circle.



So,

$$\int_{-4}^{0} \sqrt{16 - x^2}\, dx = \frac{1}{4}\pi(4)^2 = 4\pi.$$

5.3 Evaluating Definite Integrals

As we saw in the previous lesson, computing definite integrals could be nightmarishly difficult using Riemann sums. Thankfully, there is an easier way. First notice the following relationship

Theorem 5.3.1 (Evaluation/Fundamental Theorem of Calculus)**.** *If F is continuous on the interval $[a, b]$ and F is an antiderivative of f, then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof. Breaking up $[a, b]$ into n subintervals, we get that $\Delta x = \frac{b-a}{n}$. Setting $a = x_0$ and $b = x_n$, we have that

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) \\ &= F(x_n) - F(x_{n-1}) + F(x_{n-1}) - \cdots - F(x_1) + F(x_1) - F(x_0) \\ &= \sum_{i=1}^{n} [F(x_n) - F(x_{n-1})]. \end{aligned}$$

Since F is continuous (and differentiable) on each interval $[x_i, x_{i-1}]$, the Mean Value Theorem tells us that there exists x_i^* in the interval $[x_i, x_{i-1}]$ so that

$$\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = F'(x_i^*)$$

which rearranges to give us

$$\begin{aligned} F(x_i) - F(x_{i-1}) &= F'(x_i^*)(x_i - x_{i-1}) \\ F(x_i) - F(x_{i-1}) &= f(x_i^*)\Delta x. \end{aligned}$$

Summing over all these, we have that

$$F(b) - F(a) = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] = \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Taking a limit of both sides as $n \to \infty$, we have that the left-hand side is unchanged because it is constant, and the right hand side is the definition of the definite integral:

$$F(b) - F(a) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\,dx.$$

□

In lieu of the proof, let's see this theorem in action with some previous examples to confirm that it actually works.

Example 5.3.2. Use the Evaluation Theorem to find the exact value $\int_0^4 x\,dx$.

Recall from Section 4.7 that the general antiderivative of x is $\frac{1}{2}x^2 + C$. By the evaluation theorem, we have

$$\int_0^1 x\,dx = \left[\frac{1}{2}(4)^2 + C\right] - \left[\frac{1}{2}(0)^2 + C\right] = 8.$$

Indeed, this agrees with what we know about the area of a triangle of base length 4 and height 4.

Example 5.3.3. Use the Evaluation Theorem to find the exact value $\int_0^1 x^2\,dx$.

Recall from Section 4.7 that the general antiderivative of x^2 is $\frac{1}{3}x^3 + C$. By the evaluation theorem, we have that

$$\int_0^1 x^2\,dx = \left[\frac{1}{3}(1)^3 + C\right] + \left[\frac{1}{3}(0)^3 + C\right] = \frac{1}{3}.$$

This is exactly what we saw in a previous example.

These example also highlight a useful fact about using the Evaluation Theorem to compute definite integrals - we don't need the general form of the antiderivative as the added constant "$+C$" will cancel itself out in the end, so any antiderivative will do. As such, we might as well take $C = 0$.

Example 5.3.4. Find the area under the curve $y = \cos x$ over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Recall that $y = \sin x$ is an antiderivative for $\cos x$. Thus, by the Evaluation Theorem,

$$\int_{-\pi/2}^{\pi/2} \cos x\,dx = \sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right) = 1 - (-1) = 2.$$

5.3.1 Indefinite Integrals

Because of the relationship between the antiderivative and integral, we use the notation $\int f(x)\,dx$ for the general antiderivative of f, and we call this the **indefinite integral**. Specifically,

$$\int f(x)\,dx = F(x) \quad \text{is equivalent to} \quad F'(x) = f(x).$$

Remark. The definite integral $\int_a^b f(x)\,dx$ is a *number* and the indefinite integral $\int f(x)\,dx$ is a *function* (or a family of functions).

Example 5.3.5. Evaluate the following indefinite integral: $\int \left(14x^6 - \sec x \tan x\right)\, dx$.

Recall that an antiderivative for x^6 is $\frac{1}{7}x^7$ and the general antiderivative of $\sec x \tan x$ is $\sec x$. So, applying our properties of integrals,

$$\begin{aligned}\int \left(14x^6 - \sec x \tan x\right)\, dx &= \int 14x^6\, dx - \int \sec x \tan x\, dx \\ &= 14 \int x^6\, dx - \int \sec x \tan x\, dx \\ &= 14\left(\frac{1}{7}x^7\right) - (\sec x) + C \\ &= 2x^7 - \sec x + C.\end{aligned}$$

Example 5.3.6. Evaluate the following indefinite integral: $\int \frac{\sqrt{t}-1}{\sqrt{t}}\, dt$.

Recall that an antiderivative for 1 is t and an antiderivative for $t^{-1/2}$ is $-2t^{1/2}$. So, simplifying the integrand and applying properties of integrals, we have

$$\begin{aligned}\int \frac{\sqrt{t}-1}{\sqrt{t}}\, dt &= \int \left(1 - \frac{1}{\sqrt{t}}\right)\, dt \\ &= \int \left(1 - t^{-1/2}\right)\, dt \\ &= \int 1\, dt - \int t^{-1/2}\, dt \\ &= t - (-2t^{1/2}) + C \\ &= t + 2\sqrt{t} + C.\end{aligned}$$

Example 5.3.7. Suppose a particle is moving along a line with velocity function $v(t) = t^2 - 5t + 6$. Find both the displacement and distance traveled by the particle during the time interval $[1, 5]$.

The **displacement** is just the (signed) distance between the particle's starting and ending position. We can get this by simply finding the difference in the antiderivative evaluated at 1 and at 5. In other words,

$$\begin{aligned}\text{distance} &= \int_2^5 \left(t^2 - 5t + 6\right)\, dt \\ &= \left(\frac{1}{3}t^3 - \frac{5}{2}t^2 + 6t\right)\Bigg|_2^5 \\ &= \left(\frac{1}{3}(5)^3 - \frac{5}{2}(5)^2 + 6(5)\right) - \left(\frac{1}{3}(2)^3 - \frac{5}{2}(2)^2 + 6(2)\right) \\ &= \frac{9}{2} = 4.5\end{aligned}$$

Now, notice that the velocity is negative on $(2, 3)$, so the particle is back-tracking. This means that there is some distance that the particle travels in one direction that cancels out with some of the distance traveled in the opposite direction. To get the total distance traveled, we'll need to handle the negative velocity cases separately from the positive velocities. In particular, the total distance traveled is

$$\begin{aligned}
\int_2^5 \left|t^2 - 5t + 6\right| \, dt &= \int_2^3 -\left(t^2 - 5t + 6\right) \, dt + \int_3^5 \left(t^2 - 5t + 6\right) \, dt \\
&= \frac{1}{6} + \frac{14}{3} \\
&= \frac{29}{6} \approx 4.833.
\end{aligned}$$

5.4 The Fundamental Theorem of Calculus

Imagine that we have a curve $y = f(x)$ and we want to know how the area is changing if we fix the left endpoint and let the right endpoint vary. As a function (an "accumulating function"), we write

$$A(x) = \int_a^x f(t)\,dt$$

Visually, Figure 1 shows how the area is "accumulating" under the curve as we increase x.

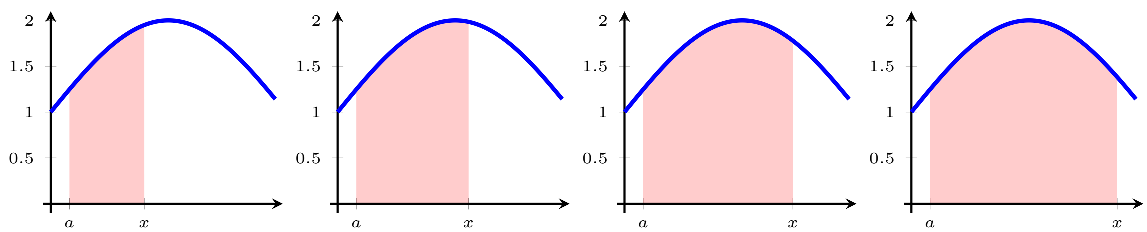
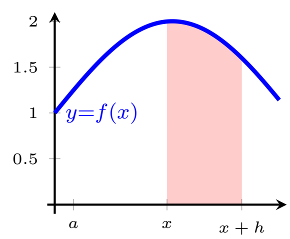


Figure 1: Area accumulating under the curve as x increases.

Example 5.4.1. Let $f(t) = 2t$, $A(x) = \int_0^x f(t)\,dt$. Find $A(0)$, $A(1)$, $A(2)$, $A(3)$, $A(4)$, and conjecture about $A(x)$.

We see that $A(0) = 0$, $A(1) = 1$, $A(2) = 4$, $A(3) = 9$, $A(4) = 16$. It seems that $A(x) = x^2$. This is interesting because $A(x)$ looks a lot like an antiderivative for f.



Notice that the difference in areas $A(x+h) - A(x)$ is approximately the area of the rectangle $h \cdot f(x)$. Well, this rearranges to

$$f(x) \approx \frac{A(x+h) - A(x)}{h}.$$

Well, as h gets smaller, the approximation gets better and better, so in fact,

$$f(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = A'(x).$$

This gives us the following

Theorem 5.4.2 (Fundamental Theorem of Calculus)**.** *If f is continuous on $[a, b]$, then the function A defined by*

$$A(x) = \int_a^x f(t)\,dt, \quad a \le x \le b,$$

is an antiderivative for f, i.e., $A'(x) = g(x)$ for $a < x < b$.

Example 5.4.3. Find $A'(x)$ given $A(x) = \int_0^x \frac{t^3+t-1}{\sqrt{t^2+7}}\,dt$.

By the Fundamental Theorem of Calculus (FTC),

$$A'(x) = \frac{x^3+x-1}{\sqrt{x^2+7}}.$$

Example 5.4.4. Find $\frac{d}{dx}[A(x)]$ where $A(x) = \int_0^x \arctan t\,dt$.

By FTC,

$$\frac{d}{dx}A(x) = \arctan x.$$

Example 5.4.5. Find $\frac{d}{dx}\int_0^{x^3} \arctan t\,dt$.

Notice that, given $A(x)$ as in the previous example, this time we're looking for $\frac{d}{dx}[A(x^3)]$. This is going to require a chain rule. Thus we have

$$\frac{d}{dx}\left[A(x^3)\right] = A'(x^3)\cdot\frac{d}{dx}\left[x^3\right] = \arctan\left(x^3\right)\cdot 3x^2.$$

Example 5.4.6. Find $\frac{d}{dx}\int_x^8 t^7\,dt$.

In order to apply FTC, we need to have the lower limit fixed and the upper limit to be a function of x. Since

$$\int_x^8 t^7\,dt = -\int_8^x t^7\,dt,$$

then we have that

$$\frac{d}{dx}\int_x^8 t^7\,dt = -\frac{d}{dx}\int_8^x t^7\,dt = -x^7$$

Of course, having a negative output on our accumulation function here makes sense. As shown in Figure 2, when the left endpoint is variable and the right endpoint is fixed, the accumulated area is decreasing as x increases.

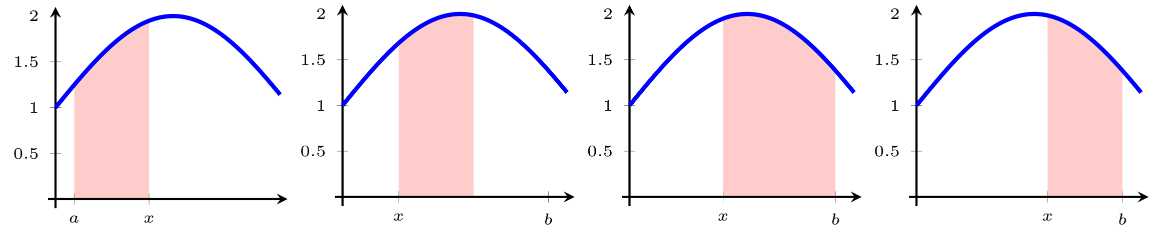


Figure 2: With the right endpoint fixed, accumulated area decreases as x increases.

Example 5.4.7. Find $\dfrac{d}{dx}\displaystyle\int_{\sin x}^{x^2+1} \sqrt{t+1}\,dt$.

We note that $\sqrt{t+1}$ has domain $[-1, \infty)$, so for any fixed number a in this interval, we have

$$\begin{aligned}\int_{\sin x}^{x^2+1} \sqrt{t+1}\,dt &= \int_{\sin x}^{a} \sqrt{t+1}\,dt + \int_{a}^{x^2+1} \sqrt{t+1}\,dt \\ &= -\int_{a}^{\sin x} \sqrt{t+1}\,dt + \int_{a}^{x^2+1} \sqrt{t+1}\,dt.\end{aligned}$$

With this new sum and applying a chain rule to each integral in the sum, we have

$$\begin{aligned}\frac{d}{dx}\int_{\sin x}^{x^2+1} \sqrt{t+1}\,dt &= -\frac{d}{dx}\int_{a}^{\sin x} \sqrt{t+1}\,dt + \frac{d}{dx}\int_{a}^{x^2+1} \sqrt{t+1}\,dt \\ &= -(\cos x)\sqrt{\sin^2 x+1} + (2x)\sqrt{4x^4+1}.\end{aligned}$$

5.4.1 Average Value of a Function

Recall that the average value of n numbers, $y_1, \ldots, y_n$ is given by

$$y_{\text{avg}} = (y_1 + y_2 + \cdots + y_n)\,\frac{1}{n}.$$

If these numbers came from a function f, so that $y_1 = f(x_1)$, $y_2 = f(x_2)$, etc., then we would have

$$f_{\text{avg}} = (f(x_1) + f(x_2) + \cdots + f(x_n))\,\frac{1}{n}.$$

Notice that this sum looks like a right Riemann sum with n rectangles over the interval $[0, 1]$. Since the x_i may instead come from the interval $[a, b]$, multiplying both sides by $(b - a)$, we get

$$\begin{aligned}&f_{\text{avg}}(b-a) = (f(x_1) + f(x_2) + \cdots + f(x_n))\left(\frac{b-a}{n}\right) = \sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}\right) \\ &\Rightarrow f_{\text{avg}} = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}\right).\end{aligned}$$

So, as we let $n \to \infty$, we get the following definition

Definition. The **average value of f on** $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example 5.4.8. Find the average value of $f(x) = 25 - x^2$ on $[0, 2]$

$$\begin{aligned} f_{\text{avg}} &= \frac{1}{2-0}\int_0^2 25 - x^2\,dx \\ &= \frac{1}{2}\left[25x - \frac{1}{3}x^3\right]_0^2 \\ &= \frac{1}{2}\left[\left(25(2) - \frac{1}{3}(2)^3\right) - \left(25(0) - \frac{1}{3}(0)^3\right)\right] \\ &= \frac{71}{3}. \end{aligned}$$

Theorem 5.4.9 (Mean Value Theorem for Integrals)**.** *If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ so that*

$$f(c) = f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Proof. Let $F(x) = \int_a^x f(t)\,dt$. Note that $F(a) = 0$. Applying the FTC and the Mean Value Theorem, there exists a c in $(0, 2)$ so that

$$\begin{aligned} f(c) &= F'(c) \\ &= \frac{F(b) - F(a)}{b-a} \\ &= \frac{1}{b-a}F(b) \\ &= \frac{1}{b-a}\int_a^b f(t)\,dt. \end{aligned}$$

□

Example 5.4.10. With the same function and interval as in Example 5.4.8. Find the value of c satisfying the Mean Value Theorem for Integrals.

We're solving for $f(c) = \frac{71}{3}$, so

$$\begin{aligned} 25 - x^2 &= \frac{71}{3} \\ c^2 &= \frac{4}{3} \\ c &= \sqrt{\frac{4}{3}} \approx 0.667. \end{aligned}$$