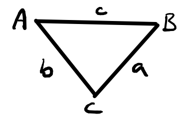
Section 6.1

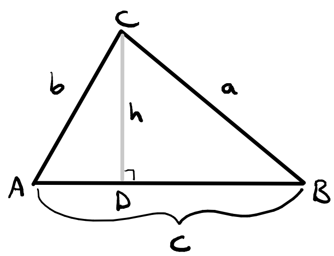
We know how to apply trig to solve right triangles, but the world is not all right triangles. Luckily, all hope is not lost.

Theorem (Law of Sines) Consider the general triangle:



Then $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

Proof



Notice that we can drop an altitude and create two right triangles. Then

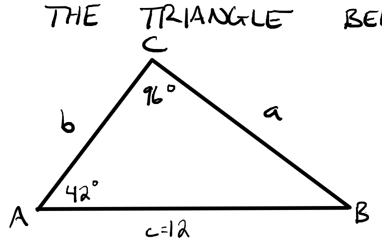
$\sin B = \dfrac{h}{a} \Rightarrow h = a\sin B$, and

$\sin A = \dfrac{h}{b} \Rightarrow h = b\sin A$. So

$$a\sin B = b\sin A$$
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

By dropping an altitude from B to $\overline{AC}$, we achieve $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$, hence the desired result. $\square$

Ex Solve the triangle below:

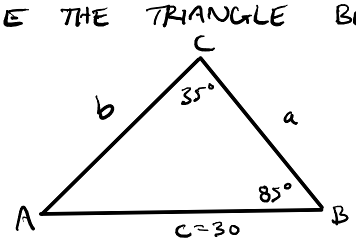


$B = 180° - 96° - 42° = 42°$

$a = \dfrac{c\sin A}{\sin C} = \dfrac{12\sin(42°)}{\sin(96°)} = 8.07$

$b = \dfrac{c\sin B}{\sin C} = \dfrac{12\sin(42°)}{\sin(96°)} = 8.07$

Ex Solve the triangle below



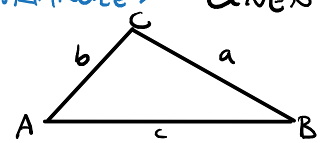
$A = 180° - 85° - 35° = 60°$

$b = \dfrac{c\sin B}{\sin C} = \dfrac{30\sin(85°)}{\sin(35°)} = 52.10$

$a = \dfrac{c\sin A}{\sin C} = \dfrac{30\sin(60°)}{\sin(35°)} = 45.30$

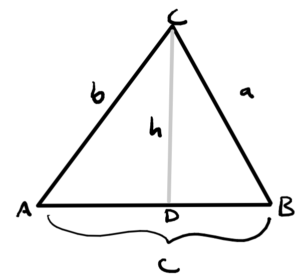
Trig also allows us to figure out the area of a general triangle.

Theorem (Area of a Triangle) Given the triangle below,



The area of this triangle is: $A = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C$.

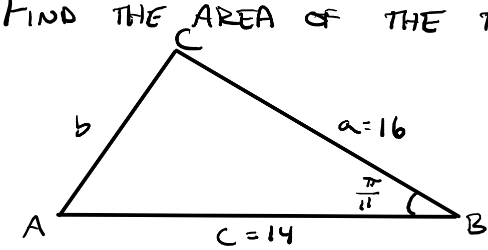
Proof Recall that the area of a triangle is given by $\frac{1}{2} \cdot \text{base length} \cdot \text{height}$.



Since $\sin A = \frac{h}{b}$, $h = b\sin A$, so

$\text{Area} = \frac{1}{2}bc\sin A$. Repeat this for the other sides. □

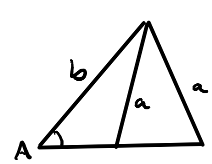
Ex Find the area of the triangle below:

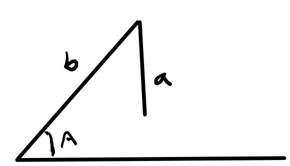


$\text{Area} = 16(14)\sin\left(\frac{\pi}{11}\right) = 63.11$

Ambiguous Triangles (SSA)

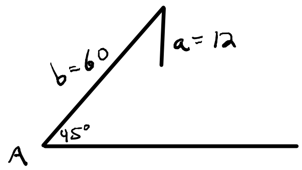
Two adjacent sides and an angle, like in the figures to the right, does not necessarily define a unique triangle.





Ex SOLVE TRIANGLE ABC, WHERE $A=45°$, $a=12$, $b=60$.

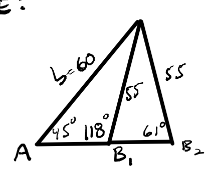
$\frac{a}{\sin A}=\frac{b}{\sin B} \Rightarrow \sin B=\frac{b\sin A}{a}=\frac{60\sin(45°)}{12}=3.53$, WHICH IS NOT IN THE RANGE OF SINE. INDEED, THE PICTURE LOOKS SOMETHING LIKE THIS



Ex SOLVE TRIANGLE ABC IF $A=45°$, $a=55$, $b=60$.

$\sin B=\frac{b\sin A}{a}=\frac{60\sin(45°)}{55}\approx 0.77$,

SO $B\approx 61.77°$ OR $118.23°$, SO EITHER PICTURE BELOW IS POSSIBLE:



IN FACT, UNLESS $B=90°$ (OR IF $a=b$, ASSUMING WE ONLY CONSIDER NONDEGENERATE CASES), THE TRIANGLE EITHER DOES NOT EXIST OR IS NOT UNIQUE.