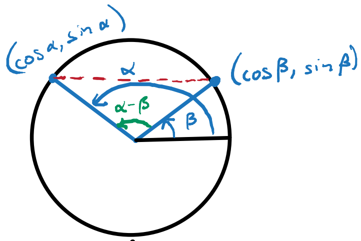
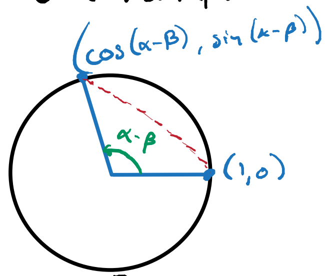
Section 5.2

Theorem (Sum/Difference Formulas) For angles α, β,

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$
$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$



A



B

Proof By the distance formula, the length of the red dashed line in figure A is

$$\begin{aligned} d &= \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2} \\ &= \sqrt{\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta + \sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta} \\ &= \sqrt{(\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)} \\ &= \sqrt{2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)}. \end{aligned}$$

The length of the red dashed line in figure B is

$$\begin{aligned} d &= \sqrt{(\cos(\alpha-\beta) - 1)^2 + (\sin(\alpha-\beta) - 0)^2} \\ &= \sqrt{\cos^2(\alpha-\beta) - 2\cos(\alpha-\beta) + 1 + \sin^2(\alpha-\beta)} \\ &= \sqrt{\cos^2(\alpha-\beta) + \sin^2(\alpha-\beta) + 1 - 2\cos(\alpha-\beta)} \\ &= \sqrt{2 - 2\cos(\alpha-\beta)} \end{aligned}$$

So we must have

$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta).$$

It follows then that

$$\begin{aligned} \cos(\alpha+\beta) = \cos(\alpha-(-\beta)) &= \cos(\alpha)\cos(-\beta) + \sin(\alpha)\sin(-\beta) \\ &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \end{aligned}$$

Recall now that $\cos(\frac{\pi}{2} - \theta) = \sin(\theta)$. So,

$$\sin(\alpha+\beta) = \cos\left(\frac{\pi}{2} - (\alpha+\beta)\right)$$
$$= \cos\left(\left(\frac{\pi}{2} - \alpha\right) - \beta\right)$$
$$= \cos\left(\frac{\pi}{2} - \alpha\right)\cos(\beta) + \sin\left(\frac{\pi}{2} - \alpha\right)\sin(\beta)$$
$$= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

AND IT FOLLOWS THAT

$$\sin(\alpha-\beta) = \sin(\alpha+(-\beta)) = \sin(\alpha)\cos(-\beta) + \cos(\alpha)\sin(-\beta)$$
$$= \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta). \qquad \square$$

THESE LEAD TO

THEOREM (SUM/DIFFERENCE RULE FOR TANGENTS)

$$\tan(\alpha \pm \beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$$

PROOF THIS IS LEFT AS AN EXERCISE FOR THE READER. IT FOLLOWS FROM $\tan\theta = \frac{\sin\theta}{\cos\theta}$ AND THE SUM/DIFFERENCE FORMULAS FOR SINE AND COSINE.

Ex $\cos(15^\circ) = \cos(60^\circ - 45^\circ)$

$$= \cos(60^\circ)\cos(45^\circ) + \sin(60^\circ)\sin(45^\circ)$$
$$= \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

Ex $\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{5\pi}{12}\right) + \cos\left(\frac{\pi}{12}\right)\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{12} + \frac{5\pi}{12}\right) = \sin\left(\frac{6\pi}{12}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

Section 5.3

Theorem (Double-Angle Formulas)

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

Proof Using the angle sum/difference formulas,

$$\sin(2\theta) = \sin(\theta+\theta) = \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta) = 2\cos(\theta)\sin(\theta).$$

The other two formulas are proven similarly. $\square$

Ex If $\sin\theta = \frac{3}{5}$ and θ in quadrant I, then $\cos\theta = \frac{4}{5}$ and so

$$\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\tfrac{3}{5}\right)\left(\tfrac{4}{5}\right) = \frac{24}{25}.$$

Notice that, applying a Pythagorean identity, we get

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1 \\ &= (1-\sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta.\end{aligned} \qquad (\star)$$

This leads us to:

Theorem (Power-Reducing Formulas)

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$
$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$
$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

Proof Rearrange the equations in $(\star)$ above. $\square$

Ex $\cos^2\left(\frac{\pi}{8}\right) = \dfrac{1+\cos\left(2\frac{\pi}{8}\right)}{2} = \dfrac{1+\cos(\pi/4)}{2} = \dfrac{1+\frac{\sqrt{2}}{2}}{2} = \dfrac{2+\sqrt{2}}{4}$

Theorem (Half-Angle Formulas)

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos(\alpha)}{2}}$$
$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos(\alpha)}{2}}$$
$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos(\alpha)}{1+\cos(\alpha)}}$$

Proof Let $\theta = \frac{\alpha}{2}$ in the power reducing formulas. $\square$

Ex $\sin(22.5^\circ) = \sin\left(\frac{45^\circ}{2}\right) = \pm\sqrt{\dfrac{1-\cos(45^\circ)}{2}} = \pm\sqrt{\dfrac{1-\sqrt{2}/2}{2}}$

$= \pm\sqrt{\dfrac{2-\sqrt{2}}{4}}$ since $\sin\theta > 0$,

$= \dfrac{\sqrt{2-\sqrt{2}}}{2}$

Section 5.4

Theorem (Product-to-Sum Formulas)

$$\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$
$$\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha-\beta) + \cos(\alpha+\beta)\right)$$
$$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$$
$$\cos\alpha\sin\beta = \frac{1}{2}\left(\sin(\alpha+\beta) - \sin(\alpha-\beta)\right)$$

Proof $\cos(\alpha-\beta) - \cos(\alpha+\beta) = (\cos\alpha\cos\beta + \sin\alpha\sin\beta) - (\cos\alpha\cos\beta - \sin\alpha\sin\beta)$

$= 2\sin\alpha\sin\beta$

The rest are proven similarly. $\square$

Theorem (Sum-to-Product Formulas)

$$\sin\alpha \pm \sin\beta = 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)$$
$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$
$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

Proof Let $\theta = \frac{\alpha+\beta}{2}$, $\psi = \frac{\alpha-\beta}{2}$. Then

$$2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) = 2\sin\theta\sin\psi$$
$$= 2\left(\frac{1}{2}\left[\sin(\theta+\psi) + \sin(\theta-\psi)\right]\right) \quad \text{(product-to-sum formula)}$$
$$= \sin(\theta+\psi) + \sin(\theta-\psi),$$
$$= \sin\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right) + \sin\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\right)$$
$$= \sin\alpha + \sin\beta.$$

The others follow similarly.