Section 3.5

Def The function $f(t) = a_0 e^{kt}$ models **exponential growth** if $k>0$, and **exponential decay** if $k<0$. k

We have already seen examples of exponential growth (compound interest) and exponential decay (radioactive decay). Let's discuss a few more.

Ex (Population Growth) Populations tend to grow exponentially. The growth model $A = 4.1e^{0.01t}$ represents New Zealand's population t years after 2006. The growth rate is 0.01. When will the population double in size? At $t=0$, i.e, in 2006, the population was 4.1 million. So we want to solve for t in the equation $8.2 = 4.1e^{0.01t}$.

$8.2 = 4.1e^{0.01t} \Rightarrow 2 = e^{0.01t} \Rightarrow \ln(2) = 0.01t \Rightarrow t \approx 69.31.$

So in the year 2075, New Zealand will double its population from 2006.

Ex Carbon dating of old objects is done by measuring the amount of carbon C-14 present in a sample, determining the amount of carbon C-14 that should have been in the sample, and then using the half-life of C-14 to figure out the sample's age. The accuracy lies in the fact that C-14's half life is about 5700 years. Its exponential decay function is $A(t) = A_0 e^{-0.000121t}$. Suppose a sample of clay from an ancient pot contains only 12% of its original C-14. Then

$0.12A_0 = A_0 e^{-0.000121t} \Rightarrow 0.12 = e^{-0.000121t} \Rightarrow \ln(0.12) = -0.000121t$, so solving for t, we see that the pot is $t \approx 16861$ year old.

Last time we talked about half-life, our model looked something like $A = A_0\left(\frac{1}{2}\right)^t$, so why does C-14 use $A = A_0 e^{-0.000121t}$? Is it special? In fact, it isn't. Any exponential model can be written as $A = A_0 e^{kt}$.

Rewriting $y = ab^x$ as an Exponential Growth/Decay Model

Since $e^{\ln(x)} = x$, $e^{\ln(b^x)} = b^x$, hence $y = ab^x = ae^{\ln(b^x)} = ae^{x\ln(b)}$.

Ex To convince us that this works, let's figure out the exponential model for carbon C-14. It has a half life of 5715 years, so

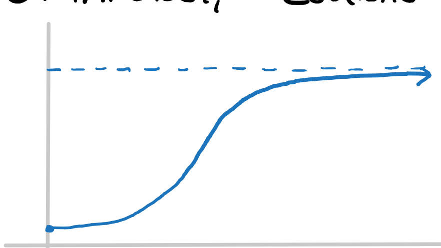
$$A = A_0\left(\tfrac{1}{2}\right)^{t/5715} = A_0 \exp\left(\frac{\ln(\frac{1}{2})t}{5715}\right) \approx A_0 e^{-0.000121t}.$$

Eureka!

Now, it is true that nothing in the real world can grow exponentially forever – eventually things like food supply will inhibit growth of a population. For this we have a different growth model.

Def A **logistic growth model** is given by $A = \dfrac{c}{1 + ae^{-bt}}$, where a, b, c are constants and $b, c > 0$. The **limiting size** is the horizontal asymptote as $t \to \infty$, which is $y = c$.

Graphically, logistic maps look like



Ex THE WORLD'S POPULATION, IN BILLIONS, t YEARS AFTER 1949, IS GIVEN BY $f(t) = \dfrac{11.82}{1 + 3.81e^{-0.027t}}$, WHICH MEANS THAT SCIENTISTS EXPECT OUR EARTH IS INCAPABLE OF SUSTAINING MORE THAN 11.82 BILLION PEOPLE.