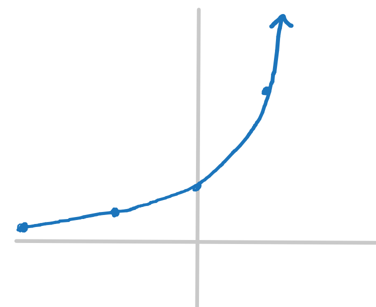
SECTION 3.1

DEF THE EXPONENTIAL FUNCTION f WITH BASE b IS
$f(x) = b^x$, WHERE $b > 0$ AND $b \neq 1$.

WHAT DOES THE GRAPH LOOK LIKE?

Ex $f(x) = 3^x$. WITH THE TABLE BELOW, THE GRAPH SHOULD LOOK SOMETHING LIKE THIS:

x	$f(x)$
-2	$3^{-2} = \frac{1}{9}$
-1	$3^{-1} = \frac{1}{3}$
0	$3^0 = 1$
1	$3^1 = 3$
2	$3^2 = 9$



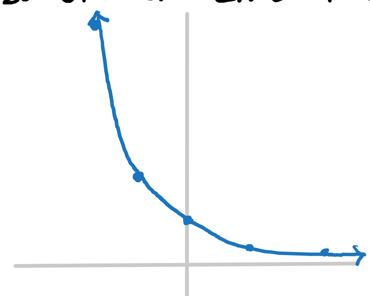
THE DOMAIN OF EXPONENTIAL FUNCTIONS IS $(-\infty, \infty)$. THIS MAY SEEM ODD, BECAUSE CERTAINLY b^k MAKES SENSE WHEN k IS AN INTEGER, AND $b^{p/q} = \sqrt[q]{b^p}$, SO IT MAKES SENSE WITH A RATIONAL EXPONENT... BUT WHAT ABOUT IRRATIONAL EXPONENTS? WELL IT TURNS OUT THAT EVERY IRRATIONAL NUMBER IS JUST SUCCESSIVELY BETTER AND BETTER APPROXIMATIONS BY RATIONAL NUMBERS. THROUGH THIS, IT TURNS OUT THAT EXPONENTIAL FUNCTIONS ARE CONTINUOUS.

REMARK: EXPONENTIAL FUNCTIONS ARE ONE-TO-ONE (THEY PASS THE HORIZONTAL LINE TEST), SO THEY HAVE INVERSES. WE'LL GET TO THESE IN THE NEXT SECTION.

Ex $g(x) = \left(\frac{1}{3}\right)^x = (3^{-1})^x = 3^{-x} = f(-x)$ (f FROM THE PREVIOUS EXAMPLE).

x	$g(x)$
-2	$\left(\frac{1}{3}\right)^{-2} = 9$
-1	$\left(\frac{1}{3}\right)^{-1} = 3$
0	$\left(\frac{1}{3}\right)^{0} = 1$
1	$\left(\frac{1}{3}\right)^{1} = \frac{1}{3}$
2	$\left(\frac{1}{3}\right)^{2} = \frac{1}{9}$

SO THE GRAPH LOOKS SOMETHING LIKE THIS:



FROM THE PRECEDING TWO EXAMPLES, WE ULTIMATELY GET THE FOLLOWING: FOR $f(x) = b^x$,

- IF $0 < b < 1$, THE GRAPH GOES UP TO THE LEFT AND IS A DECREASING FUNCTION.
- IF $b > 1$, THE GRAPH GOES UP TO THE RIGHT AND IS AN INCREASING FUNCTION.

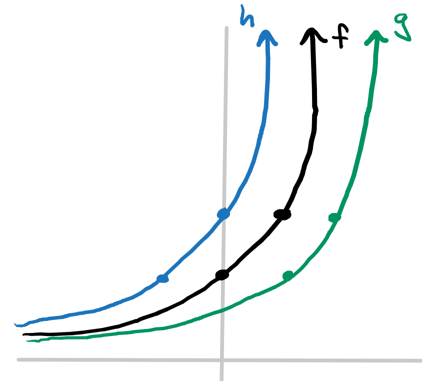
WHAT ABOUT ASYMPTOTES? NO VERTICAL ASYMPTOTES, BECAUSE DOMAIN IS $(-\infty, \infty)$. BUT WE SEE THAT AS $x \to \pm\infty$, $b^x \to$ EITHER ∞ OR 0, SO THIS MEANS WE HAVE A HORIZONTAL ASYMPTOTE $y = 0$.

WHAT ABOUT x-INTERCEPTS? IF $b > 0$, CAN b^x EVER BE ZERO? IT CANNOT, SO EXPONENTIAL FUNCTIONS HAVE NO ROOTS.

NOW, LOOKING BACK AT OUR PREVIOUS 2 EXAMPLES, $f(x) = 3^x$ AND $g(x) = \left(\frac{1}{3}\right)^x = f(-x)$ ARE JUST REFLECTIONS OF ONE ANOTHER! SO THE TRANSFORMATION OF REFLECTIONS STILL HOLDS.

Ex Let $f(x)=3^x$, $g(x)=f(x-1)=3^{x-1}$, $h(x)=f(x+1)=3^{x+1}$.

x	$f(x)$	$g(x)$	$h(x)$
-2	$\frac{1}{9}$	$3^{-2-1}=3^{-3}=\frac{1}{27}$	$3^{-2+1}=3^{-1}=\frac{1}{3}$
-1	$\frac{1}{3}$	$3^{-1-1}=3^{-2}=\frac{1}{9}$	$3^{-1+1}=3^{0}=1$
0	1	$3^{0-1}=3^{-1}=\frac{1}{3}$	$3^{0+1}=3^{1}=3$
1	3	$3^{1-1}=3^{0}=1$	$3^{1+1}=3^{2}=9$
2	9	$3^{2-1}=3^{1}=3$	$3^{2+1}=3^{3}=27$



So exponentials still obey horizontal shift transformation rules.

Playing the same game, we ultimately see that all of our transformation rules still hold.

Def The **natural base**, e, is defined to be the value that $\left(1+\frac{1}{n}\right)^n$ approaches as $n \to \infty$. The function $f(x)=e^x$ is the **natural exponential function.** In pactice, we just say "e" or "e to the x."

e is irrational, and $e \approx 2.718282$. To see why it is so "natural," we'll see what led Bernouilli to its discovery:

"Compound interest" means your interest is paid based on the amount of money in your account, not just the principal. So if you put P dollars into the bank and you earn r percent interest, compounded once per year,

Year 1: $A = P + Pr = P(1+r)$

Year 2: $A = P(1+r) + P(1+r)r = P(1+r)(1+r)^2, \ldots$

So after t-many years, you have $A = P(1+r)^t$.

Now, most institutions pay out only a fraction of their interest, but several times per year (say $\frac{r}{n}$ percent compounded n-times per year). Like this, after t-many years, you have $A = P\left(1+\frac{r}{n}\right)^{nt}$.

Now, what happens if you continually compound the interest (ie, let $n \to \infty$)? Well, letting $k = \frac{n}{r}$

$$
\begin{aligned}
A &= P\left(1+\frac{n}{r}\right)^t \\
&= P\left(1+\frac{1}{n/r}\right)^{n/r \cdot rt} \\
&= P\left(1+\frac{1}{k}\right)^{krt} \\
&= P\left[\left(1+\frac{1}{k}\right)^k\right]^{rt} \\
&= Pe^{rt}, \text{ as } k \to \infty.
\end{aligned}
$$

Let $k = n/r$. Since r is fixed, as $n \to \infty$, $k \to \infty$.

So e naturally occurs in the realm of continuously compounding interest.