Section 1.8

Def The function $f(x)=x$ is called the identity function (it is sometimes denoted $id(x)$). Much like adding 0 or multiplying by 1, this function does not change the input at all.

Most functions are not the identity function – they actually do something to the input. What we're interested in is finding a way to undo that function. More formally, given a function f, we want to find another function g s.t. $f \circ g$ is the identity function (ie $f(g(x))=x$).

Ex 1 For motivation, suppose $f(x)=x+100$. The function f adds 100 to the input. To undo adding 100, we should subtract 100. Let $g(x)=x-100$. Then

$(f\circ g)(x)=f(g(x))=f(x-100)=(x-100)+100=x,$

so g undoes f. Similarly $(g\circ f)(x)=x$

Def Let f be a function if f^{-1} is another function such that $(f^{-1}\circ f)(x)=(f\circ f^{-1})(x)=x$, then f^{-1} is called the inverse (function) of f. (f^{-1} is read "f-inverse").

Note $f^{-1}(x)$ does not mean $\frac{1}{f(x)}$!!! It is merely notation that suggests that f^{-1} and f are somehow related. -1 is not an exponent.

Proposition If f is a function and f^{-1} is an inverse function of f, then f^{-1} is unique.

Proposition $(f^{-1})^{-1}=f$.

Ex 2 Prove that $f(x) = \frac{4}{x} + 9$ and $g(x) = \frac{4}{x-9}$ are inverse functions:

$$f \circ g(x) = f(g(x)) = f\left(\frac{4}{x-9}\right)$$
$$= \frac{4}{\frac{4}{x-9}} + 9$$
$$= \frac{4(x-9)}{4} + 9$$
$$= x$$

$$(g \circ f)(x) = g(f(x))$$
$$= g\left(\frac{4}{x} + 9\right)$$
$$= \frac{4}{\left(\frac{4}{x} + 9\right) - 9}$$
$$= \frac{4}{4/x}$$
$$= \frac{4x}{4}$$
$$= x.$$

Procedure to find the inverse of a function f.

1. Replace "$f(x)$" with "y"
2. Swap x and y.
3. Solve for y.
4. Replace "y" with "$f^{-1}(x)$".

Ex 3 Let $f(x) = x^3 - 1$.

Step 1: $y = x^3 - 1$

Step 2: $x = y^3 - 1$

Step 3: $y = \sqrt[3]{x+1}$

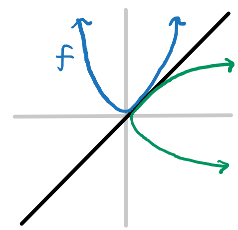
Step 4: $f^{-1}(x) = \sqrt[3]{x+1}$

Up to now, we've been glossing over the domains of f and f^{-1}! From the previous lecture, it ultimately follows that the domain of f is the range of f^{-1}, and the domain of f^{-1} is the range of f.

But what if this isn't the case?

Ex 4 Consider $f(x)=x^2$, $g(x)=\sqrt{x}$. Certainly $f\circ g(x) = g\circ f(x) = x$, but the domain of f is $(-\infty,\infty)$ and the range of g is $[0,\infty)$. What this means is that, on $(-\infty,\infty)$, f does not have an inverse, but if we restrict the domain of f to $[0,\infty)$, it does, and it is $f^{-1}(x)=\sqrt{x}$.

Graphically, the inverse of f is a reflection across the line $y=x$. In our previous example, we see that $f(x)=x^2$ is a function, but the reflection is not, because the reflected image fails the vertical line test, and by reflection, that's because $f(x)$ fails a "horizontal line test."

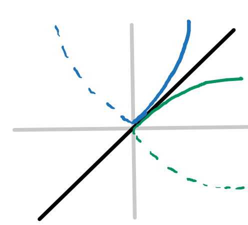


Def A function f is one-to-one (aka injective) if whenever $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Theorem (Horizontal Line Test) A function is one-to-one if and only if any horizontal line intersects the graph at most once.

Theorem A function has an inverse on a domain D if and only if it is one-to-one on the domain.

Ex 5 $f(x)=x^2$ is not one-to-one on $(-\infty,\infty)$, but it is on $[0,\infty)$.



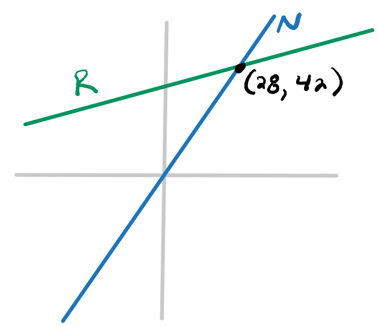
Section 1.10

Modeling real-world scenarios with math does not have any sort of one-size fits all procedure - it takes a combination of mathematical understanding and critical thinking to put together equations.

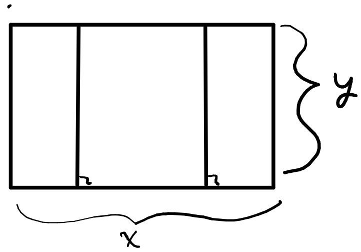
Ex 6 It costs \$1.50 per trip to ride the bus. For \$21.00 per month, you can buy a pass that reduces the bus fare to \$0.75 per trip.

The total montly cost N to ride the bus x-times at the normal rate is $N(x) = 1.5x$.
The total monthly cost R to ride the bus x-times at the reduced rate is $R(x) = 21 + 0.75x$.
The number of times you have to ride the bus in a month so that the costs are equal is

$$21 + 0.75x = 1.5x$$
$$21 = 0.75x$$
$$28 \text{ times} = x.$$



Ex 7 A farmer has 1200 ft of fencing to build the pen below:



So, $x + 4y = 1200\text{ft}$, so $y = 300 - \frac{x}{4}$. The area of the pen in terms of x is given by $A(x) = xy = x\left(300 - \frac{x}{4}\right) = -\frac{x^2}{4} + 300x$