Section 1.1

Def A relationship between two quantities can be expressed as an equation in two variables. A solution of an equation in two variables, x and y, is an ordered pair (x,y) such that when we substitute these into the equation, we get a true statement.

Ex 1 Consider the equation $y=6x-2$, and the ordered pairs $(1,4)$ and $(3,0)$. If we substitute these values for x and y in our equation, we get:

$$(4)=6(1)-2 \qquad \text{and} \qquad (0)=6(3)-2$$
$$4=4 \qquad\qquad\qquad\qquad 0=16$$

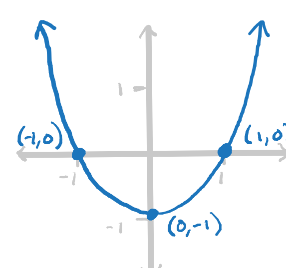
So, $(1,4)$ is a solution of the equation and $(3,0)$ is not.

Def The graph of an equation of two variables is the set of all ordered pairs that "satisfy" (ie are solutions of) the equation.

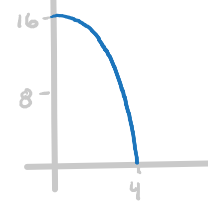
Def An x-intercept is where the graph intersects the x-axis. Note that the y-coordinate of an x-intercept is always zero, so an x-intercept always has the form $(\_,0)$.

Def A y-intercept is where the graph intersects the y-axis. Note that the x-coordinate of a y-intercept is always zero, so a y-intercept always has the form $(0,\_)$.

Ex 2 Let $y=x^2-1$. We see from the graph that the x-intercepts are $(-1,0)$ and $(1,0)$, and the y-intercept is $(0,-1)$.



Ex 3 Suppose $y = 16 - x^2$ represents the height of a ball x seconds after being dropped from a height of 16 ft. The y-intercept is $(0,16)$ and represents the height of the ball before it is dropped. An x-intercept is $(4,0)$ and represents the time at which the ball hits the ground, which is 4 seconds after being dropped.



Section 1.2

Def A relation is a set of ordered pairs.

Ex 4 The set $\{(1,0), (13,5), (7,2)\}$ is a relation.

Ex 5 The set $\{(x,y) \mid y = 2x+1\}$ (read "the set of all ordered pairs (x,y) where $y = 2x+1$") is a relation.

Def Given a relation, the set of all first components of the ordered pairs is called the domain. The set of all second components is called the range.

Ex 6 From Example 4, the domain is $\{1, 13, 7\}$ and the range is $\{0, 5, 2\}$.

Ex 7 From Example 5, the domain is $\mathbb{R}$ ("the set of all real numbers"), and the range is $\mathbb{R}$.

Def A function is a correspondence from the domain to the range, such that each element of the domain corresponds to exactly one element of the range.

WHAT THE DEFINITION OF A FUNCTION IS GETTING AT IS THAT WE ARE MAPPING/SENDING ELEMENTS OF THE DOMAIN TO ELEMENTS OF THE RANGE. MOREOVER, WE REQUIRE THAT ELEMENTS OF THE DOMAIN ARE SENT TO <u>EXACTLY ONE</u> ELEMENT IN THE RANGE. (EACH INPUT HAS ONLY ONE OUTPUT)

Ex 8 $\{(1,2),(3,4),(5,6),(5,8)\}$ IS NOT A FUNCTION BECAUSE 5 (IN THE DOMAIN) IS SENT TO BOTH 6 AND 8 (IN THE RANGE)

Ex 9 $\{(1,2),(3,4),(6,5),(8,5)\}$ IS A FUNCTION BECAUSE EACH ELEMENT OF THE DOMAIN CORRESPONDS TO ONLY ONE ELEMENT OF THE RANGE.

NOTE EXAMPLE 9 DEMONSTRATES THAT A FUNCTION CAN HAVE DIFFERENT ELEMENTS OF THE DOMAIN MAP TO THE SAME ELEMENT OF THE RANGE.

IN THIS CLASS, WE WILL BE MOSTLY CONCERNED WITH EXPRESSING FUNCTIONS AS EQUATIONS. IF WE SEE "y AS A FUNCTION OF x" OR "y IN TERMS OF x," WE MEAN $y=$(SOME EXPRESSION INVOLVING x).

Ex 10 LET $w=14d-7$. w IS A FUNCTION OF d SINCE THERE IS EXACTLY ONE w-VALUE CORRESPONDING TO EACH d-VALUE.

Ex 11 LET $y^2=x^2+1$. TAKING SQUARE ROOTS OF BOTH SIDES GIVES US $y=\pm\sqrt{x^2+1}$, EACH VALUE OF x CORRESPONDS TO 2 VALUES OF y, y IS NOT A FUNCTION OF x.

A FUNCTION CAN BE THOUGHT OF AS A KIND OF MACHINE, CALL IT f, TAKING INPUTS x FROM THE DOMAIN AND OUTPUTTING $f(x)$ IN THE RANGE

Ex 12 LET $f(x) = x^3 - 5$, READ AS "f OF x EQUALS $x^3 - 5$." FOR EACH INPUT x, $f(x)$ IS THE VALUE OF THE FUNCTION AT x. IF, SAY, WE WANTED TO FIND $f(1)$ AND $f(5)$, WE SIMPLY SUBSTITUTE IN FOR x IN THE RIGHT-HAND-SIDE OF THE EQUATION.

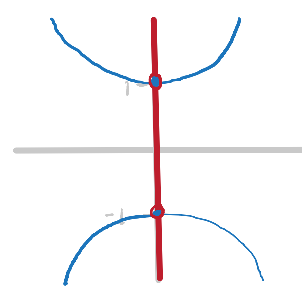
$$f(1) = (1)^3 - 5 = -4, \qquad f(5) = (5)^3 - 5 = 120.$$

THIS IS CALLED FUNCTION NOTATION.

Ex 13 SUPPOSE f, AS IN EXAMPLE 12, REPRESENTS THE POPULATION OF TEMPE x YEARS AFTER 1990. THEN $f(1)$ IS THE POPULATION IN 1991, AND $f(5)$ IS THE POPULATION IN 1995. IF b IS SOME OTHER VARIABLE, THEN $f(b+3)$ IS THE POPULATION b YEARS AFTER 1993.

DEF A GRAPH OF A FUNCTION f IS THE SET OF ORDERED PAIRS $(x, f(x))$ PLOTTED ON THE CARTESIAN PLANE. A GRAPH IS NOT A FUNCTION, BUT RATHER A PICTORIAL REPRESENTATION OF A FUNCTION.

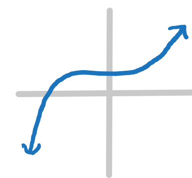
Ex 14 (VERTICAL LINE TEST) LET'S PLOT $y^2 = x^2 + 1$ FROM EXAMPLE 11. WE ALREADY DETERMINED THIS WAS NOT A FUNCTION, BUT WE CAN ALSO SEE THIS FROM THE GRAPH. AT $x = 0$, $y = \pm 1$, SO THE VERTICAL LINE $x = 0$ INTERSECTS THE GRAPH IN TWO PLACES. THIS LEADS TO THE FOLLOWING RESULT.



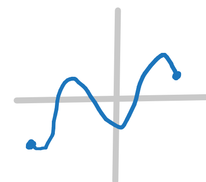
VERTICAL LINE TEST IF ANY VERTICAL LINE INTERSECTS A GRAPH IN MORE THAN ONE POINT, THE GRAPH DOES NOT DEFINE y AS A FUNCTION OF f.

When we look at the graph of a function,

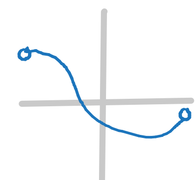
- Arrows indicate that the graph extends indefinitely in the direction of the arrows.



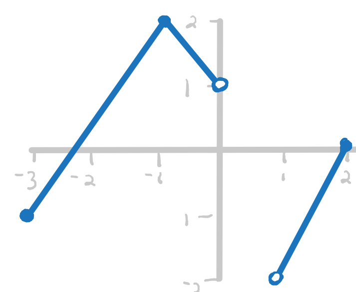
- A closed dot indicates that the graph does not extend past the point and it <u>includes</u> that point



- An open circle indicates that the graph does not extend past the point and it <u>excludes</u> that point.



Ex 14 Using the graph on the right, we see that the function has



	Interval notation	Set-builder notation
Domain	$[-3,0)\cup(1,2]$	$\{x \mid -3\le x<0 \text{ or } 1<x\le 2\}$
Range	$(-2,2]$	$\{y \mid -2<y\le 2\}$

Def The zeroes of a function f are the x-values for which $f(x)=0$.

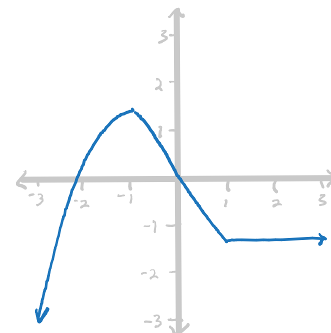
When looking at the graph of a function f, zeroes of a function have the form $(x,0)$, which means they correspond to x-intercepts. Thus we can find the x-intercepts of a function's graph without ever graphing it.

Since y-intercepts of a graph have the form $(0,\_)$, the y-intercept of the graph of a function is the point $(0,f(0))$.

SECTION 1.3

DEF A FUNCTION IS INCREASING ON AN OPEN INTERVAL (a,b) IF $f(x_1) < f(x_2)$ WHENEVER $a < x_1 < x_2 < b$. A FUNCTION IS DECREASING ON AN OPEN INTERVAL (a,b) IF $f(x_1) > f(x_2)$ WHENEVER $a < x_1 < x_2 < b$. A FUNCTION IS CONSTANT ON AN OPEN INTERVAL IF $f(x_1) = f(x_2)$ WHENEVER $a < x_1 < x_2 < b$.

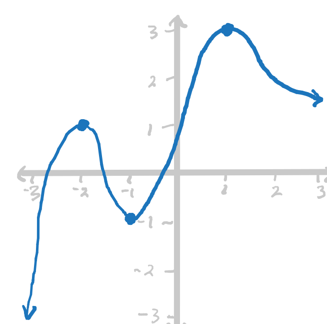
Ex 1 THE FUNCTION TO THE RIGHT IS INCREASING ON $(-\infty, -1)$, DECREASING ON $(-1, 1)$, AND CONSTANT ON $(1, \infty)$.



DEF A FUNCTION VALUE $f(a)$ IS A RELATIVE MAXIMUM OF f IF THERE EXISTS AN OPEN INTERVAL CONTAINING a SUCH THAT $f(a) > f(x)$ FOR ALL $x \neq a$ IN THE OPEN INTERVAL.

2) A FUNCTION VALUE $f(b)$ IS A RELATIVE MINIMUM OF f IF THERE EXISTS AN OPEN INTERVAL CONTAINING b SUCH THAT $f(b) < f(x)$ FOR ALL $x \neq b$ IN THE OPEN INTERVAL.

Ex 2 THE FUNCTION IN THE GRAPH TO THE RIGHT HAS RELATIVE MAXIMA AT -2 AND 1, AND THE RELATIVE MAXIMA ARE $f(-2) = 1$ AND $f(1) = 3$, RESPECTIVELY. THE FUNCTION HAS A RELATIVE MINIMUM AT -1, AND THE RELATIVE MINIMUM IS $f(-1) = -1$.



DEF A FUNCTION IS EVEN IF $f(-x) = f(x)$ FOR ALL x IN THE DOMAIN.
A FUNCTION IS ODD IF $f(-x) = -f(x)$ FOR ALL x IN THE DOMAIN.

Ex 3 $f(x) = x^4 - 2$ IS EVEN BECAUSE $f(-x) = (-x)^4 - 2 = x^4 - 2 = f(x)$.
$g(x) = 2x\sqrt{x^2-1}$ IS ODD BECAUSE $g(-x) = 2(-x)\sqrt{(-x)^2-1} = -2x\sqrt{x^2-1} = -g(x)$.

THE NEXT DEFINITION WILL BE IMPORTANT SOON (AND AGAIN IN CALCULUS), BUT FOR NOW FEELS A BIT UNMOTIVATED.

DEF THE EXPRESSION $\dfrac{f(x+h)-f(x)}{h}$ FOR $h \neq 0$ IS CALLED THE DIFFERENCE QUOTIENT OF THE FUNCTION f.

Ex 4 GIVEN $f(x) = 3x^2 + x$, THE DIFFERENCE QUOTIENT FOR $h \neq 0$ IS

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} = \frac{3(x+h)^2+(x+h)-(3x^2+x)}{h} &= \frac{3(x^2+2hx+h^2)+x+h-3x^2-x}{h} \\ &= \frac{3x^2+6hx+3h^2+x+h-3x^2-x}{h} \\ &= \frac{3h^2+6hx+h}{h} \\ &= 3h+6x+1\end{aligned}$$

SECTION 1.4

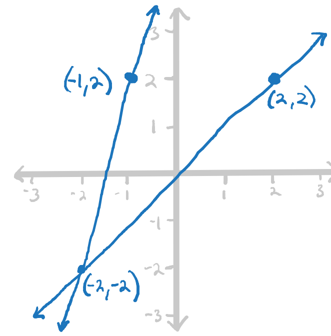
DEF THE SLOPE OF A LINE BETWEEN TWO POINTS (x_1, y_1) AND (x_2, y_2) IS $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. THE SLOPE IS A RATIO, "RISE OVER RUN," AND AND IS A MEASURE OF THE STEEPNESS OF A LINE.

NOTE: WHEN $x_1 = x_2$ AND $y_1 \neq y_2$, THE LINE IS VERTICAL AND THE SLOPE IS UNDEFINED. IN THIS CASE THE LINE IS NOT A FUNCTION.

Section 1.4 (Continued)

Ex 5 The slope of the line through $(-2,-2)$ and $(2,2)$ is $\frac{(2)-(-2)}{(2)-(-2)} = \frac{2+2}{2+2} = \frac{4}{4} = 1$.

The slope of the line through $(-2,-2)$ and $(-1,2)$ is $\frac{(2)-(-2)}{(-1)-(-2)} = \frac{2+2}{-1+2} = \frac{4}{1} = 4$.



Def The point-slope form of the equation of the line with slope m through point (x_1, y_1) is $(y-y_1) = m(x-x_1)$.

Def The slope-intercept form of the equation of the line with slope m with y-intercept $(0,b)$ is $y = mx+b$.

Ex 6 The line passing through $(1,-2)$ with slope -3 has point-slope form $(y+2) = -3(x-1)$.

The line with y-intercept $(0,1)$ with slope -3 has slope-intercept form $y = -3x+1$.

Def The equation of a horizontal line through $(0,b)$ is $y=b$.

The equation of a vertical line through $(a,0)$ is $x=a$.

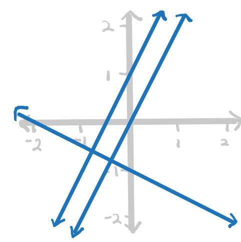
Def Every line has an equation that can be written in the general form $Ax+By+C=0$, where A, B, C are real numbers, and A and B are not both zero.

Section 1.5

Def Two lines in the same plane are **parallel** if they don't intersect. A handy feature about parallel lines is that they have the same slope.

Def Two lines are **perpendicular** if they intersect at a right (90°) angle. A handy feature of perpendicular lines is that their slopes are negative reciprocals of one another.

Ex 1 $y=2x$ and $y=2x+1$ are parallel lines. $y=2x$ and $y=-\frac{1}{2}x-1$ are perpendicular lines.



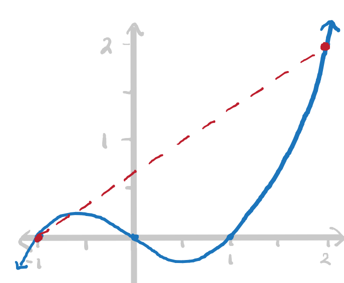
Def The **average rate of change** between any two points is the slope of the line containing the two points. This line is called the **secant line.**

Note: We <u>do not</u> calculate the average rate of change by finding an average. It is <u>not</u> the average of two points.

The average rate of change of a function f from x_1 to x_2 is given by $\dfrac{f(x_2)-f(x_1)}{x_2-x_1}$.

Ex 2 Let $f(x)=\frac{1}{3}x^3-\frac{1}{3}x$

Rate of change from $(-1,0)$ to $(2,2)$ is $\dfrac{2-0}{2-(-1)}=\dfrac{2}{3}$.



Section 1.6

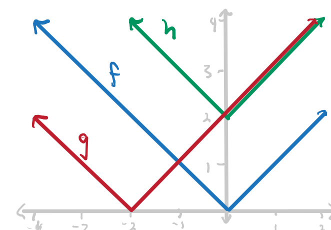
Def A parent function is one without any sort of transformation, like $f(x)=x$, $f(x)=|x|$, $f(x)=x^2$, and so on.

Transformations can affect a function vertically or horizontally.

- Horizontal transformations occur when we modify the function's input
- Vertical transformations occur when we modify the function's output.

Def A shift occurs when we add/subtract a constant.

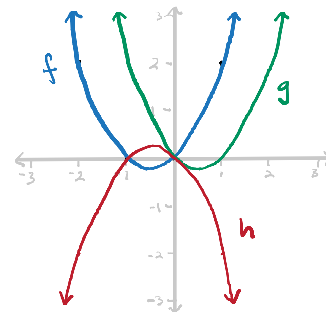
Ex Consider the parent function $f(x)=|x|$. $g(x)=f(x+2)=|x+2|$ shifts the graph left by 2 units. $h(x)=f(x)+2=|x|+2$ shifts the graph up by 2 units.



In short, for a function $f(x)$ and $c>0$,
$f(x+c)$ shifts left, $f(x-c)$ shifts right,
$f(x)+c$ shifts up, $f(x)-c$ shifts down.

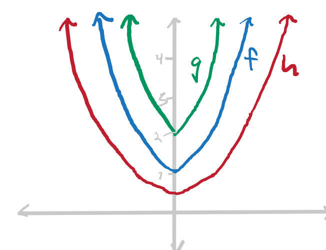
Def A reflection occurs when we multiply by -1.

Ex 3 Consider the parent function $f(x) = x^2 + x$. Then $g(x) = f(-x) = x^2 - x$ is a reflection about the y-axis and $h(x) = -f(x) = -x^2 - x$ is a reflection about the x-axis.

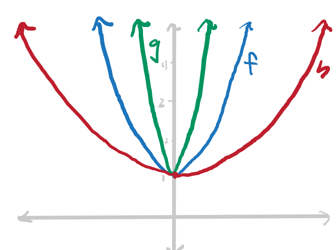


Def A **stretch** or **shrink** occurs when we multiply by a positive constant $c \neq 1$.

Ex 4 (Vertical Stretch / Shrink) Consider the parent function $f(x) = x^2 + 1$. Then $g(x) = 2f(x) = 2x^2 + 2$ is a vertical stretch and $h(x) = \frac{1}{2} f(x) = \frac{1}{2}x^2 + \frac{1}{2}$ is a vertical shrink.



Ex 5 (Horizontal Strech / Shrink) Consider the parent function $f(x) = x^2 + 1$ Then $g(x) = f(2x) = 4x^2 + 1$ is a horizontal shrink and $h(x) = f\left(\frac{1}{2}x\right) = \frac{1}{4}x^2 + 1$ is a horizontal stretch.



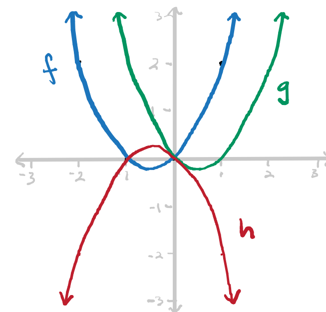
In short, for a function f and a constant c,

When $c > 1$, $cf(x)$ is a vertical stretch and $f(cx)$ is a horizontal shrink.

When $0 < c < 1$, $cf(x)$ is a vertical shrink and $f(cx)$ is a horizontal stretch.

When graphing a function with a sequence of transformations, preform them in the following order: 1) Horizontal shift, 2) Reflection, 3) stretching/shrinking, and 4) vertical shift.

Ex 3 Consider the parent function $f(x) = x^2 + x$. Then $g(x) = f(-x) = x^2 - x$ is a reflection about the y-axis and $h(x) = -f(x) = -x^2 - x$ is a reflection about the x-axis.



Def A stretch or shrink occurs when we multiply by a positive constant $c \neq 1$.

Ex 4 (Vertical Stretch/Shrink) Consider the parent function $f(x) = x^2$. Then $g(x) = 2f(x) = 2x^2$ is a vertical stretch and $h(x) = \frac{1}{2}f(x) = \frac{1}{2}x^2$ is a vertical shrink.

Ex 5 (Horizontal Strech/Shrink) Consider the parent function $f(x) = x^3$. Then $g(x) = f(2x) = 8x^3$ is a horizontal shrink and $h(x) = f(\frac{1}{2}x) = \frac{1}{8}x^3$ is a horizontal stretch.

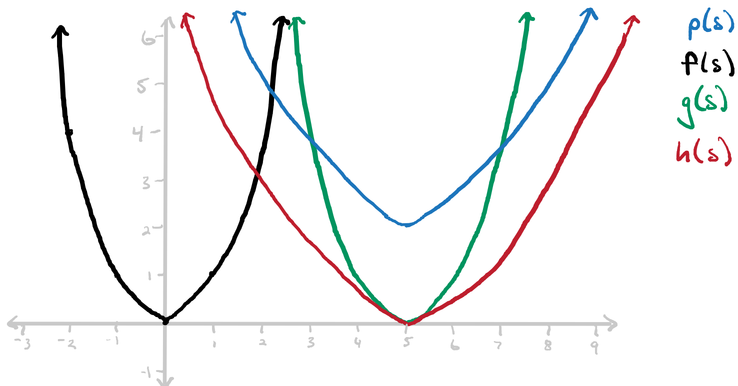
In short, for a function f and a constant c,

When $c > 1$, $cf(x)$ is a vertical stretch and $f(cx)$ is a horizontal shrink.

When $0 < c < 1$, $cf(x)$ is a vertical shrink and $f(cx)$ is a horizontal stretch.

When graphing a function with a sequence of transformations, preform them in the following order: 1) Horizontal shift, 2) Reflection, 3) stretching/shrinking, and 4) vertical shift.

Ex 6 Say we want to graph the function $p(t) = \frac{1}{3}(t-5)^2 + 2$. Start with the parent function $f(t) = t^2$. Then graph $g(t) = (t-5)^2$, the horizontal shift right. Then graph $h(t) = \frac{1}{3}(t-5)^2$, the vertical shrink. Then graph $p(t) = \frac{1}{3}(t-5)^2 + 2$, the vertical shift.

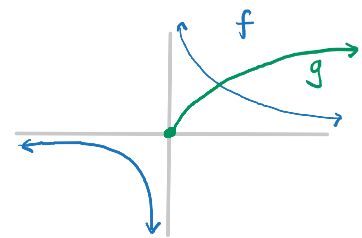


Section 1.7

Before we saw the notion of the domain of a relation and determined the domain based on graphs. Since a function is just a special type of relation,

Def The **domain** of a function $f(x)$ is the set of all x-values where $f(x)$ is a real number.

Ex 1 $f(x) = \frac{1}{x}$ has domain $(-\infty, 0) \cup (0, \infty)$.
$g(x) = \sqrt{x}$ has domain $[0, \infty)$.



Let f, g be two functions w/ domains D_f, D_g, respectively. Let $D = D_f \cap D_g$ be the set of all elements common to both D_f and D_g. The **sum** $f+g$, **difference** $f-g$, **product** fg, and **quotient** f/g are functions with domains D defined as follows:

1. sum $(f+g)(x) = f(x) + g(x)$
2. difference $(f-g)(x) = f(x) - g(x)$
3. product $(fg)(x) = f(x)g(x)$
4. quotient $(f/g)(x) = f(x)/g(x)$, provided $g(x) \neq 0$, so in fact f/g may have more restrictions on its domain.

Ex 2 Let $f(x) = x+1$, $g(x) = x^2 - 4$. Both have domains $(-\infty, \infty)$. Then $(f+g)(x) = x^2 + x - 3$. $(f-g)(x) = -x^2 + x + 5$. $(fg)(x) = (x+1)(x^2-4) = x^3 + x^2 - 4x - 4$. $(f/g)(x) = \frac{x+1}{x^2-4}$. The first 3 have domain $(-\infty, \infty)$. The quotient has domain $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

NOTE: DETERMINE THE DOMAIN __BEFORE__ SIMPLIFYING.

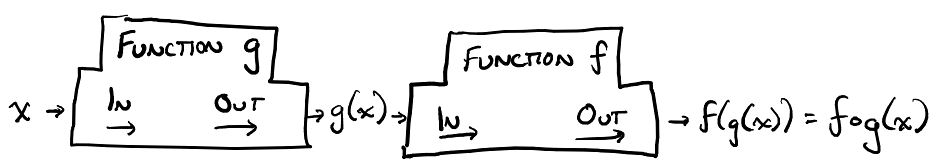
Ex 3 LET $f(x) = x^2 - 1 = (x+1)(x-1)$, $g(x) = x - 1$.

$\left(f/g\right)(x) = \dfrac{x^2-1}{x-1}$, SO f/g HAS DOMAIN $(-\infty, 1) \cup (1, \infty)$, BUT

$= \dfrac{(x+1)(x-1)}{x-1} = x + 1.$

THE **COMPOSITION OF THE FUNCTION f WITH g** IS THE FUNCTION $f \circ g$ (READ "f COMPOSED WITH g") DEFINED BY $(f \circ g)(x) = f(g(x))$. THE **DOMAIN OF $f \circ g$** IS THE SET OF ALL x SUCH THAT

1. x IS IN THE DOMAIN OF g, AND
2. $g(x)$ IS IN THE DOMAIN OF f.



Ex 4 LET $f(x) = x^2 + 1$, $g(x) = x + 1$. THEN

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f(x+1) \\ &= (x+1)^2 + 1 \\ &= x^2 + 2x + 2 \end{aligned} \qquad \begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g(x^2+1) \\ &= (x^2+1) + 1 \\ &= x^2 + 2 \end{aligned}$$

NOTE: $f \circ g$ AND $g \circ f$ NEED NOT BE THE SAME FUNCTION!

Ex 5 LET $f(x) = x + 1$, $g(x) = \sqrt{x}$. THEN

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f(\sqrt{x}) \\ &= \sqrt{x} + 1 \end{aligned} \qquad \begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g(x+1) \\ &= \sqrt{x+1} \end{aligned}$$

DOMAIN: $[0, \infty)$ DOMAIN: $[-1, \infty)$.

AS WE SAW WITH TRANSFORMATIONS, IT IS USEFUL TO THINK OF FUNCTIONS AS COMPOSITIONS OF SIMPLER FUNCTIONS, WE CALL THIS DECOMPOSITION.

Ex LET $h(x) = (\sqrt{x}+7)^9$. WE CAN WRITE $h(x) = f\circ g(x)$, WHERE $f(x) = x^9$ AND $g(x) = \sqrt{x}+7$. WE CAN ALSO CHOOSE $f(x) = (x+1)^7$ AND $g(x) = \sqrt{x}$.

DECOMPOSITIONS ARE NOT NECESSARILY UNIQUE.

Section 1.8

Def The function $f(x) = x$ is called the identity function (it is sometimes denoted $id(x)$). Much like adding 0 or multiplying by 1, this function does not change the input at all.

Most functions are not the identity function – they actually do something to the input. What we're interested in is finding a way to undo that function. More formally, given a function f, we want to find another function g s.t. $f \circ g$ is the identity function (ie $f(g(x)) = x$).

Ex 1 For motivation, suppose $f(x) = x + 100$. The function f adds 100 to the input. To undo adding 100, we should subtract 100. Let $g(x) = x - 100$. Then

$$(f \circ g)(x) = f(g(x)) = f(x-100) = (x-100)+100 = x,$$

so g undoes f. Similarly $(g \circ f)(x) = x$

Def Let f be a function if f^{-1} is another function such that $(f^{-1} \circ f)(x) = (f \circ f^{-1})(x) = x$, then f^{-1} is called the inverse (function) of f. (f^{-1} is read "f-inverse").

Note $f^{-1}(x)$ does not mean $\frac{1}{f(x)}$!!! It is merely notation that suggests that f^{-1} and f are somehow related. -1 is not an exponent.

Proposition If f is a function and f^{-1} is an inverse function of f, then f^{-1} is unique.

Proposition $(f^{-1})^{-1} = f$.

Ex 2 PROVE THAT $f(x) = \frac{4}{x} + 9$ AND $g(x) = \frac{4}{x-9}$ ARE INVERSE FUNCTIONS:

$$f \circ g(x) = f(g(x)) = f\left(\frac{4}{x-9}\right)$$
$$= \frac{4}{\frac{4}{x-9}} + 9$$
$$= \frac{4(x-9)}{4} + 9$$
$$= x$$

$$(g \circ f)(x) = g(f(x))$$
$$= g\left(\frac{4}{x} + 9\right)$$
$$= \frac{4}{\left(\frac{4}{x} + 9\right) - 9}$$
$$= \frac{4}{4/x}$$
$$= \frac{4x}{4}$$
$$= x.$$

PROCEDURE TO FIND THE INVERSE OF A FUNCTION f.

1. REPLACE "$f(x)$" WITH "y"
2. SWAP x AND y.
3. SOLVE FOR y.
4. REPLACE "y" WITH "$f^{-1}(x)$".

Ex 3 LET $f(x) = x^3 - 1$.

STEP 1: $y = x^3 - 1$

STEP 2: $x = y^3 - 1$

STEP 3: $y = \sqrt[3]{x+1}$

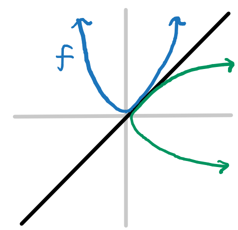
STEP 4: $f^{-1}(x) = \sqrt[3]{x+1}$

UP TO NOW, WE'VE BEEN GLOSSING OVER THE DOMAINS OF f AND f^{-1}! FROM THE PREVIOUS LECTURE, IT ULTIMATELY FOLLOWS THAT THE DOMAIN OF f IS THE RANGE OF f^{-1}, AND THE DOMAIN OF f^{-1} IS THE RANGE OF f.

BUT WHAT IF THIS ISN'T THE CASE?

Ex 4 CONSIDER $f(x)=x^2$, $g(x)=\sqrt{x}$. CERTAINLY $f\circ g(x)=g\circ f(x)=x$, BUT THE DOMAIN OF f IS $(-\infty,\infty)$ AND THE RANGE OF g IS $[0,\infty)$. WHAT THIS MEANS IS THAT, ON $(-\infty,\infty)$, f DOES NOT HAVE AN INVERSE, BUT IF WE RESTRICT THE DOMAIN OF f TO $[0,\infty)$, IT DOES, AND IT IS $f^{-1}(x)=\sqrt{x}$.

GRAPHICALLY, THE INVERSE OF f IS A REFLECTION ACROSS THE LINE $y=x$. IN OUR PREVIOUS EXAMPLE, WE SEE THAT $f(x)=x^2$ IS A FUNCTION, BUT THE REFLECTION IS NOT, BECAUSE THE REFLECTED IMAGE FAILS THE VERTICAL LINE TEST, AND BY REFLECTION, THAT'S BECAUSE $f(x)$ FAILS A "HORIZONTAL LINE TEST."

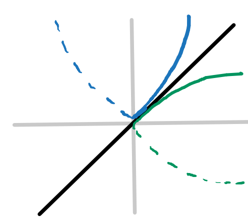


DEF A FUNCTION f IS ONE-TO-ONE (AKA INJECTIVE) IF WHENEVER $x_1\neq x_2$, THEN $f(x_1)\neq f(x_2)$.

THEOREM (HORIZONTAL LINE TEST) A FUNCTION IS ONE-TO-ONE IF AND ONLY IF ANY HORIZONTAL LINE INTERSECTS THE GRAPH AT MOST ONCE.

THEOREM A FUNCTION HAS AN INVERSE ON A DOMAIN D IF AND ONLY IF IT IS ONE-TO-ONE ON THE DOMAIN.

Ex 5 $f(x)=x^2$ IS NOT ONE-TO-ONE ON $(-\infty,\infty)$, BUT IT IS ON $[0,\infty)$.



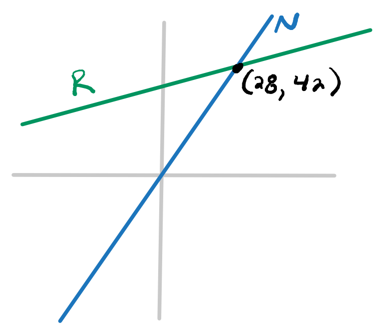
Section 1.10

Modeling real-world scenarios with math does not have any sort of one-size fits all procedure - it takes a combination of mathematical understanding and critical thinking to put together equations.

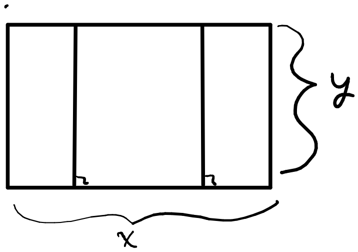
Ex 6 It costs \$1.50 per trip to ride the bus. For \$21.00 per month, you can buy a pass that reduces the bus fare to \$0.75 per trip.

The total montly cost N to ride the bus x-times at the normal rate is $N(x) = 1.5x$.
The total monthly cost R to ride the bus x-times at the reduced rate is $R(x) = 21 + 0.75x$.
The number of times you have to ride the bus in a month so that the costs are equal is

$$21 + 0.75x = 1.5x$$
$$21 = 0.75x$$
$$28 \text{ times} = x.$$



Ex 7 A farmer has 1200 ft of fencing to build the pen below:



So, $x + 4y = 1200\text{ft}$, so $y = 300 - \frac{x}{4}$. The area of the pen in terms of x is given by $A(x) = xy = x\left(300 - \frac{x}{4}\right) = -\frac{x^2}{4} + 300x$

Section 2.1

Def The imaginary unit is defined to be $i=\sqrt{-1}$, where $i^2=-1$.

Def A complex number is a number of the form $a+bi$, where a,b are real numbers and i is the imaginary unit. This is called the standard form for complex numbers, a is called the real part, and b is called the imaginary part.

Ex 1 $\sqrt{-81}=\sqrt{(-1)81}=\sqrt{i^2\cdot 9^2}=9i$
$\sqrt{-3}=\sqrt{(-1)3}=\sqrt{i^2\cdot 3}=i\sqrt{3}$,

Generally, to avoid confusion, we write i before a square root, so $3+6i\sqrt{2}$ is still in standard form.

Proposition Let $a+bi$, $c+di$ be complex numbers. Then $a+bi=c+di$ if and only if $a=c$ and $b=d$.

When adding/subtracting complex numbers, we treat the real and imaginary parts separately.

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$
$$(a+bi)-(c+di)=(a-c)+(b-d)i$$

Ex 2 $(11-2i)+(6+3i)=(11+6)+(-2+3)i=17+i$
$(11-2i)-(6+3i)=(11-6)+(-2-3)i=5-5i$

Since complex numbers are basically polynomials in i instead of x, they multiply in the same way:

$(a+bi)(c+di)=ac+bci+adi+bdi^2=(ac-bd)+(ad+bc)i$, since $i^2=1$.

Ex 3 $(3-4i)(5+7i) = 3\cdot 5 - 5\cdot 4i + 3\cdot 7i - 4\cdot 7i^2$
$= 15 - 20i + 21i + 28 = 43 + i$

DEF GIVEN A COMPLEX NUMBER $a+bi$, THE COMPLEX CONJUGATE OF $a+bi$ IS THE COMPLEX NUMBER $a-bi$. A HANDY FEATURE OF THE COMPLEX CONJUGATE IS THAT $(a+bi)(a-bi)$ IS A REAL NUMBER.

$$(a+bi)(a-bi) = a^2 + abi - abi - b^2 i^2 = a^2 + b^2.$$

DIVIDING BY COMPLEX NUMBERS REQUIRES THE USE OF THE COMPLEX CONJUGATE.

$$\frac{c+di}{a+bi} = \frac{c+di}{a+bi}\cdot\left(\frac{a-bi}{a-bi}\right) = \frac{(c+di)(a-bi)}{(a+bi)(a-bi)} = \frac{(c+di)(a-bi)}{a^2+b^2}$$
$$= \frac{(ac-bd)+(ad+bc)i}{a^2+b^2}$$
$$= \left(\frac{ac-bd}{a^2+b^2}\right) + \left(\frac{ad+bc}{a^2+b^2}\right)i$$

Ex 4 $$\frac{2+3i}{5-2i} = \frac{(2+3i)(5+2i)}{(5+2i)(5-2i)} = \frac{10+15i+4i-6}{25+4} = \frac{4+19i}{29} = \frac{4}{29} + \frac{19}{29}i$$

DEF GIVEN A REAL NUMBER $b>0$, THE PRINCIPAL SQUARE ROOT OF $-b$ IS DEFINED TO BE $\sqrt{-b} = i\sqrt{b}$

NOTE THE PRODUCT RULE FOR RADICALS ONLY APPLIES WHEN THE RADICANDS ARE POSITIVE. WHEN PERFORMING OPERATIONS INVOLVING SQUARE ROOTS OF NEGATIVE NUMBERS, REWRITE THEM IN TERMS OF i FIRST

THE REASONS FOR THIS ARE DEEPLY ROOTED IN COMPLEX ANALYSIS, SO FOR NOW IT WILL HAVE TO REMAIN SOMEWHAT MYSTERIOUS. BUT HERE IS AN EXAMPLE OF WHY IT MATTERS.

Ex 5 RIGHT: $\sqrt{-6}\cdot\sqrt{-6} = i\sqrt{6}\cdot i\sqrt{6} = i^2\sqrt{36} = -6$

WRONG: $\sqrt{-6}\cdot\sqrt{-6} = \sqrt{(-6)^2} = \sqrt{36} = 6$

AND CERTAINLY $6 \neq -6$!

THE REASON WE EVEN THOUGHT ABOUT COMPLEX NUMBERS IN THE FIRST PLACE IS BECAUSE WE WANTED TO FIND ROOTS OF THE POLYNOMIAL x^2+1. THIS LED TO US FINDING ROOTS OF MORE GENERAL QUADRATIC POLYNOMIALS.

RECALL: FOR THE EQUATION $ax^2+bx+c=0$, WITH $a\neq 0$, THE QUADRATIC FORMULA GIVES THE SOLUTIONS

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

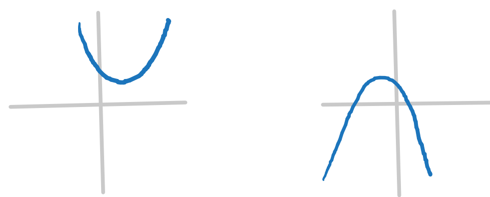
Ex 6 THE SOLUTIONS $3x^2-5x+3=0$ ARE

$$\begin{aligned} x &= \frac{5 \pm \sqrt{25-4(3)(3)}}{2(3)} \\ &= \frac{5 \pm \sqrt{25-36}}{6} \\ &= \frac{5 \pm \sqrt{-11}}{6} \\ &= \frac{5}{6} \pm i\frac{\sqrt{11}}{6}. \end{aligned}$$

SECTION 2.2

DEF A QUADRATIC FUNCTION IS A FUNCTION OF THE FORM $f(x)=ax^2+bx+c$, WHERE $a \neq 0$.

IF $a>0$, THE PARABOLA "OPENS UPWARD." IF $a<0$, IT "OPENS DOWNWARD."



$a>0$ $a<0$

DEF THE VERTEX (OF THE PARABOLA) IS THE LOWEST POINT ON THE GRAPH WHEN IT OPENS UPWARD AND IS THE HIGHEST POINT ON THE GRAPH WHEN IT OPENS DOWNWARD

DEF THE VERTICAL LINE THROUGH THE VERTEX IS CALLED THE AXIS OF SYMMETRY.

DEF THE STANDARD FORM FOR A QUADRATIC FUNCTION IS $f(x)=a(x-h)^2+k$, WHERE $a \neq 0$.

THIS MAY SEEM STRANGE AT FIRST, BUT IT'S A PARTICULARLY USEFUL FORM AS IT ALLOWS US TO JUST READ OFF THE VERTEX AND AXIS OF SYMMETRY. IN PARTICULAR, THE VERTEX IS AT THE POINT (h,k), AND THE AXIS OF SYMMETRY IS $x=h$.

SINCE PARABOLAS HAVE THIS NICE SYMMETRY, IT MEANS WE CAN MORE EASILY AND MORE ACCURATELY GRAPH THEM.

To graph $f(x) = a(x-h)^2 + k$

1. Determine if it opens upward or downward.
2. Determine vertex and axis of symmetry.
3. Find any x-intercepts, solving $f(x) = 0$ (real zeroes only).
4. Find any y-intercepts, $f(0)$.
5. Plot it!

Ex 1 $f(x) = 2(x+1)^2 - 2 = 2(x^2+2x+1) - 2 = 2x(x+2)$.

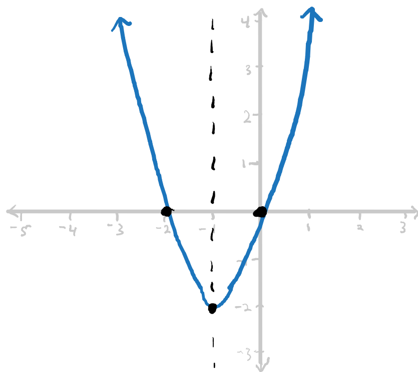
2) vertex at $(-1,-2)$, axis of symmetry is $x = -2$.

1) $2 > 0$, so opens up.

3) $2x(x+2) = 0 \Rightarrow x = -2, 0$ are x-intercepts.

4) $f(0) = 0$ is y-intercept.

5)



Is it possible to put every quadratic function into this standard form? Yep, via a method called "completing the square," (which is actually how you prove the quadratic formula).

Completing the Square

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x\right)+c$$
$$= a\left(x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2\right)+c$$
$$= a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)-\frac{b^2}{4a}+c$$
$$= a\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a}+c$$

Ex 2 $f(x) = 3x^2-12x+1 = 3(x^2-4x)+1$

$$= 3\left(x^2-4x+\left(\frac{4}{2}\right)^2-\left(\frac{4}{2}\right)^2\right)+1$$
$$= 3\left(x^2-4x+(-2)^2-(-2)^2\right)+1$$
$$= 3(x^2-4x+2^2)-12+1$$
$$= 3(x-2)^2-11$$

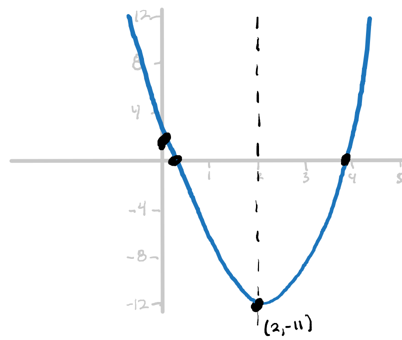
VERTEX: $(2,-11)$

AXIS OF SYMMETRY: $x=2$

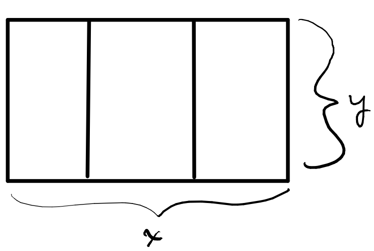
y-INTERCEPT: 1

OPENS UP

x-INTERCEPTS: $2 \pm \frac{1}{3}\sqrt{33}$



Ex 3 Remember our farmer from two days ago? He had 1200 ft of fencing to build the pen below:



We deduced that $A(x) = -\frac{x^2}{4}+300x$ represented the area as a function of side length. Now the farmer wants to maximize the area. Since the parabola opens down, the max occurs at the vertex: $(600, 90\,000)$

$A(x) = -\frac{x^2}{4}+300x = -\frac{1}{4}(x^2-1200x) = -\frac{1}{4}(x^2-1200x+600^2-600^2) = -\frac{1}{4}(x-600)^2+300^2$

When the width is 600 ft, his pen encloses 90,000 ft.

Section 2.3

Def A **polynomial function** is a function of the form $f(x) = a_n x^n + \cdots + a_1 x + a_0$, where $a_0, \ldots, a_n$ are real numbers. This is called a **polynomial of degree n** and a_n is called the **leading coefficient**.

Note: For polynomial functions, n must be a nonnegative integer.

Ex 4 $f(x) = x^3 + 5$, $g(x) = 3$, $h(x) = 16x - 13x^{19} + 8 - 31x^2$ are polynomial functions.

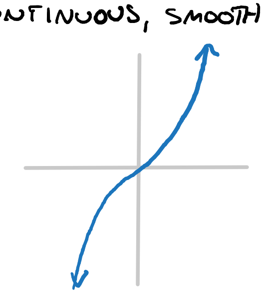
$p(x) = x^{-1} = \frac{1}{x}$, $q(x) = x^{1/2} - 3x^2 = \sqrt{x} - 3x^2$, $r(x) = 2^x$ are not polynomial functions.

Def A function is **continuous** if its graph can be drawn w/o lifting a pencil off the paper. A function is **smooth** if it has no sharp corners.

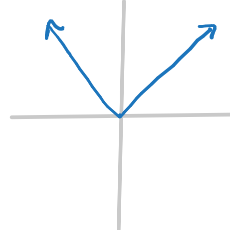
(These definitions will work for this class, but it should be noted that they are way too loose to be actual mathematical definitions)

Ex 5 $f(x) = x^3$ | $g(x) = |x|$ | $h(x) = \frac{x^2(x-1)}{x-1}$

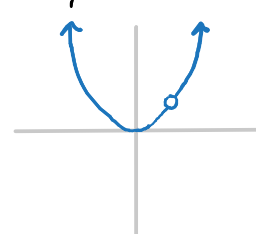
Continuous, smooth



Continuous, not smooth



Smooth, not continuous



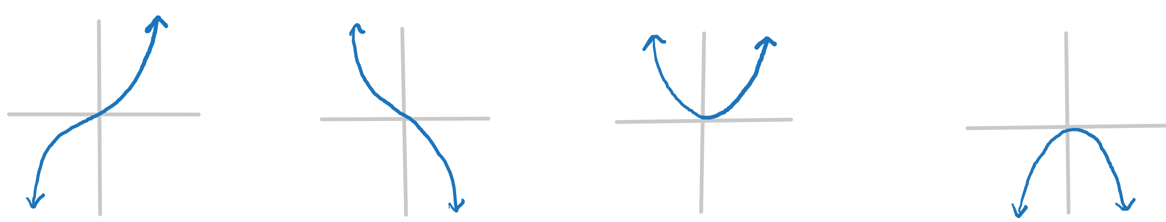
Result: Polynomial functions are smooth and continuous.

Def The **end behavior** of the graph of a function is a description of what happens to the far left and far right of the graph (secretly, as x heads off to $-\infty$ and ∞, resp.)

Theorem (Leading Coefficient Test) Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$, $a_n \neq 0$, be a polynomial function. We have four cases:

1) If n is odd and $a_n > 0$, the function falls to the left and rises to the right $(\swarrow \nearrow)$
2) If n is odd and $a_n < 0$, the function rises to the left and falls to the right $(\nwarrow \searrow)$.
3) If n is even and $a_n > 0$, the function rises to both the left and right $(\nwarrow \nearrow)$
4) If n is even and $a_n < 0$, the function falls to both the left and right $(\swarrow \searrow)$

Ex 6 $f(x) = x^3$ $\qquad$ $f(x) = -x^3$ $\qquad$ $f(x) = x^2$ $\qquad$ $f(x) = -x^2$



Ex 7 $f(x) = -31x^{7962186351} + x^2 + 385{,}000{,}000{,}002$

By the leading coefficient test, this rises to the left and falls to the right.

Section 2.3 (Cont'd)

Def A **zero** (or **root** or **solution**) of a polynomial function f is is the x-value s.t. $f(x)=0$.

Finding roots is a highly nontrivial procedure, and there is no one-size-fits-all approach.

Ex 1 $f(x)=x^2-9=(x+3)(x-3)=0 \Rightarrow f$ has roots $x=\pm 3$.

$g(x)=x^3-16x-x^2+16=x(x^2-16)-(x^2-16)=(x-1)(x^2-16)=(x-1)(x-4)(x+4)=0$

$\Rightarrow g$ has roots $x=1,\pm 4$.

Remark: Real roots of a polynomial function correspond to x-intercepts on the graph.

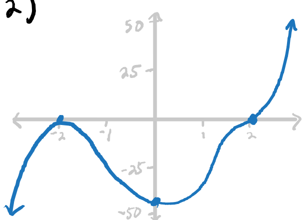
Def Let r be a root of $f(x)$. If $g(x)$ is a function such that $f(x)=(x-r)^k g(x)$ and $g(r)\neq 0$, the integer k is the **multiplicity of the root r**.

Ex $f(x)=x^4(x-1)^3$. 0 is a root of multiplicity 4. 1 is a root of multiplicity 3.

Def A root of multiplicity 1 is called a **simple root**; a root of multiplicity $k>1$ is called a **multiple root**.

Result If r is a root of even multiplicity touches the graph touches the x-axis at r and turns around. If it has odd multiplicity, it crosses the x-axis at r.

Ex $f(x) = (x+2)^2(x-2)^3$



Theorem (Intermediate Value) Let f be a polynomial function. If $f(a)$ and $f(b)$ have different signs (say $f(a)>0$, $f(b)<0$), then there exists a c s.t. $a<c<b$ and $f(c)=0$.

Ex $f(x) = x^3 - x - 1$. Since $f(1) = -1$ and $f(2) = 5$, f has a root in the interval $(1,2)$.

Section 2.4

Theorem (Division Algorithm) Let $d(x)$, $f(x)$ be polynomials with $d(x) \neq 0$ and $\deg d(x) \leq \deg f(x)$. Then there exist unique polynomials $q(x)$ and $r(x)$ s.t. $f(x) = d(x)\,q(x) + r(x)$. If $r(x)$, the **remainder** is zero, we say $d(x)$ **divides evenly** into $f(x)$, and that $d(x)$, $q(x)$ are **factors** of $f(x)$.

Ex (Polynomial Long Division) $(x^4 + 2x - 1) \div (x^2 + 3x) = x^2 - 3x + 9 + \frac{-25x-1}{x^2+3x}$

$$
\begin{array}{r|l}
 & x^2 - 3x + 9 \\
\hline
x^2 + 3x & x^4 + 0x^3 + 0x^2 + 2x - 1 \\
 & -(x^4 + 3x^3) \\
\hline
 & -3x^3 + 0x^2 \\
 & -(-3x^3 - 9x^2) \\
\hline
 & 9x^2 + 2x \\
 & -(9x^2 + 27x) \\
\hline
 & -25x - 1
\end{array}
$$

Theorem (Remainder Theorem) If $f(x)$ is divided by $(x-c)$, the remainder is $f(c)$.

This allows us to solve for $f(c)$ w/o ever plugging in $f(c)$.

Ex $f(x)=2x^3-11x^2+7x-5$, find $f(4)$

$$
\begin{array}{r|l}
 & 2x^2-3x-5 \\ \hline
x-4 & 2x^3-11x^2+7x-5 \\
 & -(2x^3-8x^2) \downarrow \\ \cline{2-2}
 & -3x^2+7x \\
 & -(-3x^2+12x) \downarrow \\ \cline{2-2}
 & -5x-5 \\
 & -(-5x+20) \\ \cline{2-2}
 & -25 = f(4)
\end{array}
$$

Theorem (Factor Theorem) $(x-c)$ is a factor of $f(x)$ if and only if $f(c)=0$.

Section 2.5

Theorem (Rational Root Theorem) If $f(x)=a_nx^n+\cdots+a_1x+a_0$ has integer coefficents and p/q (in lowest terms) is a rational root of f, then p is a factor of a_0 and q is a factor of a_n.

Ex $2x^3-5x^2+x+6$

Factors of constant term: $\pm 1, \pm 2, \pm 3, \pm 6$

Factors of leading term: $\pm 1, \pm 2$

Possible rational roots: $\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{6}{1}, \pm\frac{1}{2}, \pm\frac{3}{2}$.

Section 2.6

Def Rational functions are quotients of polynomial functions $f(x) = p(x)/q(x)$. The domain is the set of all x-values for which the denominator is not zero.

Ex $f(x) = \dfrac{x^3-5x+7}{(x-1)(x-3)}$ has domain $(-\infty,1)\cup(1,3)\cup(3,\infty)$.
or $\{x \mid x \neq 1,3\}$.

Arrow Notation

$x \to a^+$ means "x approaches a from the right."

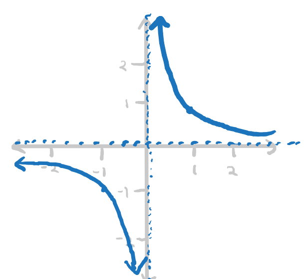
$x \to a^-$ means "x approaches a from the left."

$x \to \infty$ means "x increases without bound."

$x \to -\infty$ means "x decreases without bound."

This notation is useful because it allows us to talk about what happens near a point where a function is undefined.

Ex $\dfrac{1}{x^3}$ has domain $(-\infty,0)\cup(0,\infty)$.
As $x \to 0^+$, $f(x) \to \infty$. We typically write $\lim_{x\to 0^+} f(x) = \infty$. Similarly, $\lim_{x\to 0^-} f(x) = -\infty$, and $\lim_{x\to\infty} f(x) = 0$.

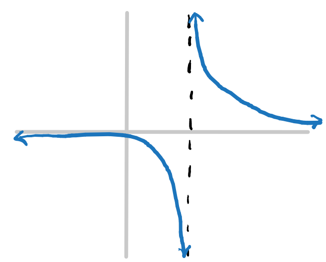


Def The vertical line $x=a$ is a vertical asymptote of f if $f(x)$ increases or decreases w/o bound as x approaches a.

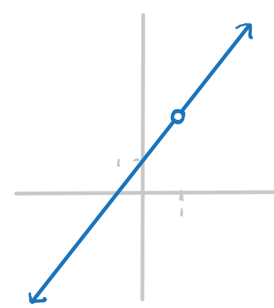
Note: If $x=a$ is a vertical asymptote of f, the graph of f <u>never crosses the line $x=a$</u>.

THEOREM LET $f(x) = p(x)/q(x)$ BE A RATIONAL FUNCTION. IF $p(x)$ AND $q(x)$ HAVE NO COMMON FACTORS AND IF a IS A ROOT OF q, THEN $x=a$ IS A VERTICAL ASYMPTOTE.

Ex $r(x) = \dfrac{1}{x-1}$. THE NUMERATOR AND DENOMINATOR HAVE NO COMMON FACTORS, SO $x=1$ IS A VERTICAL ASYMPTOTE.



Ex $s(x) = \dfrac{x^2-1}{x-1}$. THE NUMERATOR AND DENOMINATOR DO HAVE A COMMON FACTOR, AND IT IS $x-1$, SO $x=1$ IS NOT A VERTICAL ASYMPTOTE. BUT THE DOMAIN IS $(-\infty,1)\cup(1,\infty)$ AND ON THIS DOMAIN $s(x) = \frac{x^2-1}{x-1} = x+1$, SO $s(x)=x+1$ WITH A POINT MISSING AT $x=1$.



DEF IF $f(x) = \dfrac{p(x)}{q(x)}$ AND $p(a)=q(a)=0$, $f(x)$ HAS A **HOLE** AT $x=a$.

DEF A **HORIZONTAL ASYMPTOTE** OCCURS WHEN $\lim_{x\to\infty} f(x) = b$ OR $\lim_{x\to-\infty} f(x) = c$ (OR BOTH), WHERE b AND c ARE (POSSIBLY THE SAME) REAL NUMBER. THE HORIZONTAL ASYMPTOTES ARE THE LINES $y=b$ AND $y=c$.

NOTE: IF $y=b$ IS A HORIZONTAL ASYMPTOTE OF f, THE GRAPH OF f <u>MAY CROSS THE LINE $y=b$.</u>

Ex $\lim_{x\to\infty} \frac{x^2}{x} = \lim_{x\to\infty} \frac{x}{1} = \infty$

$\lim_{x\to-\infty} \frac{x^2}{x} = \lim_{x\to-\infty} \frac{x}{1} = -\infty$

$f(x) = \frac{x^2}{x}$ HAS NO HORIZONTAL ASYMPTOTES.

Ex $\lim_{x\to\infty} \frac{2x}{3x} = \lim_{x\to\infty} \frac{2}{3} = \frac{2}{3}$

$\lim_{x\to-\infty} \frac{2x}{3x} = \lim_{x\to\infty} \frac{2}{3} = \frac{2}{3}$

$g(x) = \frac{2x}{x}$ HAS A HORIZONTAL ASYMPTOTE OF $y = \frac{2}{3}$

Ex $\lim_{x\to\infty} \frac{x}{x^2} = \lim_{x\to\infty} \frac{1}{x} = 0$

$\lim_{x\to-\infty} \frac{x}{x^2} = \lim_{x\to-\infty} \frac{1}{x} = 0$

$h(x) = \frac{x}{x^2}$ HAS A HORIZONTAL ASYMPTOTE OF $y=0$.

PROPOSITION LET $f(x) = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$ BE A RATIONAL FUNCTION.

1) IF $n > m$, THEN f HAS NO HORIZONTAL ASYMPTOTES.
2) IF $n = m$, THEN f HAS A HORIZONTAL ASYMPTOTE $y = \frac{a_n}{b_m}$.
3) IF $n < m$, THEN f HAS A HORIZONTAL ASYMPTOTE $y = 0$.

PROCEDURE TO PLOT RATIONAL FUNCTIONS $f(x) = \frac{p(x)}{q(x)}$

1. DETERMINE IF f HAS SYMMETRY (ie IS EVEN, ODD, NEITHER)
2. FIND ANY y-INTERCEPTS.
3. FIND ANY x-INTERCEPTS.
4. FIND ANY HOLES AND/OR VERTICAL ASYMPTOTES.
5. FIND ANY HORIZONTAL ASYMPTOTES.
6. PLOT AT LEAST ONE POINT TO THE LEFT AND RIGHT OF EACH ROOT AND VERTICAL ASYMPTOTE.

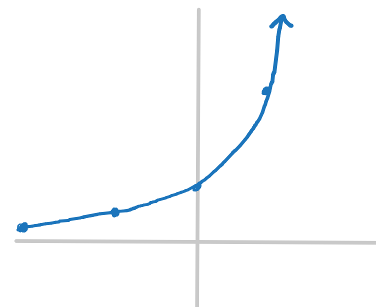
Section 3.1

Def The exponential function f with base b is $f(x) = b^x$, where $b > 0$ and $b \neq 1$.

What does the graph look like?

Ex $f(x) = 3^x$. With the table below, the graph should look something like this:

x	$f(x)$
-2	$3^{-2} = \frac{1}{9}$
-1	$3^{-1} = \frac{1}{3}$
0	$3^0 = 1$
1	$3^1 = 3$
2	$3^2 = 9$



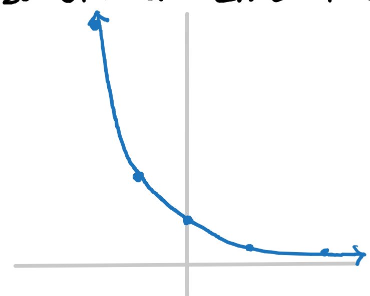
The domain of exponential functions is $(-\infty, \infty)$. This may seem odd, because certainly b^k makes sense when k is an integer, and $b^{p/q} = \sqrt[q]{b^p}$, so it makes sense with a rational exponent... but what about irrational exponents? Well it turns out that every irrational number is just successively better and better approximations by rational numbers. Through this, it turns out that exponential functions are continuous.

Remark: Exponential functions are one-to-one (they pass the horizontal line test), so they have inverses. We'll get to these in the next section.

Ex $g(x) = \left(\frac{1}{3}\right)^x = \left(3^{-1}\right)^x = 3^{-x} = f(-x)$ (f FROM THE PREVIOUS EXAMPLE).

x	$g(x)$
-2	$\left(\frac{1}{3}\right)^{-2} = 9$
-1	$\left(\frac{1}{3}\right)^{-1} = 3$
0	$\left(\frac{1}{3}\right)^{0} = 1$
1	$\left(\frac{1}{3}\right)^{1} = \frac{1}{3}$
2	$\left(\frac{1}{3}\right)^{2} = \frac{1}{9}$

SO THE GRAPH LOOKS SOMETHING LIKE THIS:



FROM THE PRECEDING TWO EXAMPLES, WE ULTIMATELY GET THE FOLLOWING: FOR $f(x) = b^x$,

- IF $0 < b < 1$, THE GRAPH GOES UP TO THE LEFT AND IS A DECREASING FUNCTION.
- IF $b > 1$, THE GRAPH GOES UP TO THE RIGHT AND IS AN INCREASING FUNCTION.

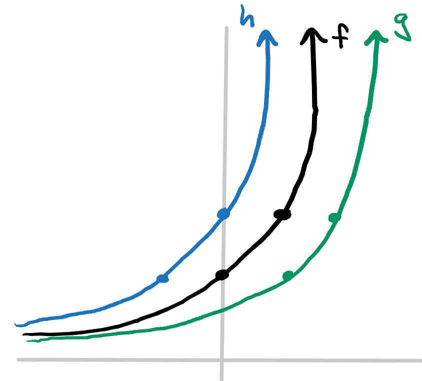
WHAT ABOUT ASYMPTOTES? NO VERTICAL ASYMPTOTES, BECAUSE DOMAIN IS $(-\infty, \infty)$. BUT WE SEE THAT AS $x \to \pm\infty$, $b^x \to$ EITHER ∞ OR 0, SO THIS MEANS WE HAVE A HORIZONTAL ASYMPTOTE $y = 0$.

WHAT ABOUT x-INTERCEPTS? IF $b > 0$, CAN b^x EVER BE ZERO? IT CANNOT, SO EXPONENTIAL FUNCTIONS HAVE NO ROOTS.

NOW, LOOKING BACK AT OUR PREVIOUS 2 EXAMPLES, $f(x) = 3^x$ AND $g(x) = \left(\frac{1}{3}\right)^x = f(-x)$ ARE JUST REFLECTIONS OF ONE ANOTHER! SO THE TRANSFORMATION OF REFLECTIONS STILL HOLDS.

Ex Let $f(x)=3^x$, $g(x)=f(x-1)=3^{x-1}$, $h(x)=f(x+1)=3^{x+1}$.

x	$f(x)$	$g(x)$	$h(x)$
-2	$\frac{1}{9}$	$3^{-2-1}=3^{-3}=\frac{1}{27}$	$3^{-2+1}=3^{-1}=\frac{1}{3}$
-1	$\frac{1}{3}$	$3^{-1-1}=3^{-2}=\frac{1}{9}$	$3^{-1+1}=3^0=1$
0	1	$3^{0-1}=3^{-1}=\frac{1}{3}$	$3^{0+1}=3^1=3$
1	3	$3^{1-1}=3^0=1$	$3^{1+1}=3^2=9$
2	9	$3^{2-1}=3^1=3$	$3^{2+1}=3^3=27$



So exponentials still obey horizontal shift transformation rules.

Playing the same game, we ultimately see that all of our transformation rules still hold.

Def The natural base, e, is defined to be the value that $\left(1+\frac{1}{n}\right)^n$ approaches as $n \to \infty$. The function $f(x)=e^x$ is the natural exponential function. In pactice, we just say "e" or "e to the x."

e is irrational, and $e \approx 2.718282$. To see why it is so "natural," we'll see what led Bernouilli to its discovery:

"Compound interest" means your interest is paid based on the amount of money in your account, not just the principal. So if you put P dollars into the bank and you earn r percent interest, compounded once per year,

Year 1: $A = P + Pr = P(1+r)$

Year 2: $A = P(1+r) + P(1+r)r = P(1+r)(1+r)^2, \ldots$

So after t-many years, you have $A = P(1+r)^t$.

Now, most institutions pay out only a fraction of their interest, but several times per year (say $\frac{r}{n}$ percent compounded n-times per year). Like this, after t-many years, you have $A = P\left(1+\frac{r}{n}\right)^{nt}$.

Now, what happens if you continually compound the interest (i.e., let $n \to \infty$)? Well, letting $k = \frac{n}{r}$

$$
\begin{aligned}
A &= P\left(1+\frac{n}{r}\right)^{t} \\
&= P\left(1+\frac{1}{n/r}\right)^{n/r \cdot rt} \\
&= P\left(1+\frac{1}{k}\right)^{krt} \\
&= P\left[\left(1+\frac{1}{k}\right)^{k}\right]^{rt} \\
&= Pe^{rt}, \text{ as } k \to \infty.
\end{aligned}
$$

Let $k = n/r$. Since r is fixed, as $n \to \infty$, $k \to \infty$.

So e naturally occurs in the realm of continuously compounding interest.

Section 3.2

We said last time that, for $b>0$ with $b\neq 1$, the function $f(x)=b^x$ has an inverse.

Step 1: $y=b^x$

Step 2: $x=b^y$

Step 3: ?

We don't have a way to handle this. So, we create new notation:

Def For $x\neq 0$, $b>0$, and $b\neq 1$,

$$y=\log_b(x) \text{ is equivalent to } x=b^y.$$

$f(x)=\log_b(x)$ is the **logarithmic function with base b.**

Ex $3=\log_5 x \Rightarrow 5^3=x \Rightarrow x=125$

$y=\log_2 64 \Rightarrow 2^y=64 \Rightarrow y=6$

$3=\log_b 27 \Rightarrow b^3=27 \Rightarrow b=3$

Basic Logarithm Properties: Let $b>0$ and $b\neq 1$. Then

1) $\log_b(b)=1$, since $b^1=b$.
2) $\log_b(1)=0$, since $b^0=1$

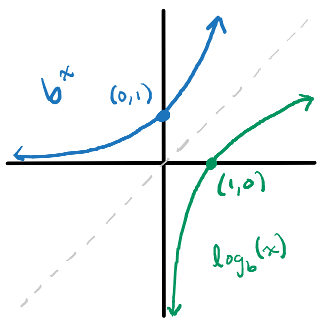
We said that, for $f(x)=b^x$, $f^{-1}(x)=\log_b(x)$. Since $f^{-1}(f(x))=x$ and $f(f^{-1}(x))=x$,

Inverse Property For $b>0$ and $b\neq 1$, we have

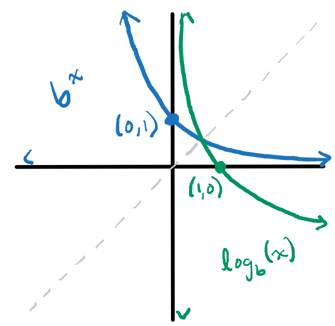
1) $f(f^{-1}(x))=f(\log_b(x))=b^{\log_b(x)}=x$, and
2) $f^{-1}(f(x))=f^{-1}(b^x)=\log_b(b^x)=x$.

Since $f(x)=b^x$ has domain $(-\infty,\infty)$ and range $(0,\infty)$, $f^{-1}(x)=\log_b(x)$ has domain $(0,\infty)$ and range $(-\infty,\infty)$.

For $b>1$



For $0<b<1$



Claim All transformations of functions still hold for logarithmic functions.

Def The logarithm with base 10 is the **common logarithm**. It is often denoted as just $\log(x)$, without the base. The logarithm with base e is the **natural logarithm**. It is often denoted $\ln(x)$. This is the convention we'll adhere to in our class.

Note: WolframAlpha and other higher-level math texts use $\log(x)$ or $\text{Log}(x)$ to denote the natural logarithm.

Ex The Richter scale, which measures earthquake intensity, is logarithmic. So if an earthquake is 10^k times as strong, it registers a $\log(10^k)=\log_{10}(10^k)=k$ on the Richter scale.

Ex Sound intensity in deciBels is also measured on a base-10 logarithmic scale (see exercises 117 + 118 in Section 3.2).

Section 3.3

Let $b>0$, $b\neq 1$, and let $M,N>0$ be real numbers. Let $x=\log_b(M)$, $y=\log_b(N)$, so then $M=b^x$, $N=b^y$.

$$MN=b^x b^y=b^{x+y},$$

so then

$$\log_b(MN)=\log_b(b^{x+y})=x+y=\log_b(M)+\log_b(N).$$

Product Rule for Logarithms. For $b,M,N>0$ with $b\neq 1$, we have $\log_b(MN)=\log_b(M)+\log_b(N)$

With b,M,N,x,y as before, we have

$$M/N=b^x/b^y=b^{x-y},$$

so

$$\log_b(M/N)=\log_b(b^{x-y})=x-y=\log_b M-\log_b(N).$$

Quotient Rule for Logarithms. For $b,M,N>0$ with $b\neq 1$, we have $\log_b\left(\frac{M}{N}\right)=\log_b(M)-\log_b(N)$.

With b,M as before, and p any nonzero real number. Set $x=\log_b(M^p)$ so that $b^x=M^p$. Then $b^{x/p}=(b^x)^{1/p}=(M^p)^{1/p}=M^{p/p}=M$,

so

$$\log_b(M)=\log_b(b^{x/p})=\frac{x}{p}=\frac{1}{p}\log_b(M^p),$$
$$\Rightarrow \log_b(M^p)=p\log_b(M).$$

We only proved this for nonzero p, but the result is obviously true for $p=0$ (why?).

Power Rule for Logarithms Let $b,M>0$, $b\neq 1$, and p is any real number. Then $\log_b(M^p)=p\log_b(M)$.

Section 3.3 (Continued)

Using properties of logarithms, we can also condense expressions.

Ex $\ln(18x-2) - \ln(y) - \ln(41) = \ln\left(\frac{18x-2}{41y}\right)$ (use PEMDAS)

Ex $\log_2(3) - \frac{1}{2}\log_2(x+15) = \log_2\left(\frac{3}{\sqrt{x+15}}\right)$

We have one more property that will prove extremely useful when explicitly computing logarithms. First, let $a, b > 0$ with $a, b \neq 1$, and let $M > 0$. Set $x = \log_a(M)$ and $y = \log_b(M)$, so that $M = a^x = b^y$. Then

$$\begin{aligned} y = \log_b(M) &= \log_b(a^x) \\ &= x \log_b(a) \\ &= \log_a(M)\log_b(a). \end{aligned}$$

$$\Rightarrow \log_a(M) = \frac{\log_b(M)}{\log_b(a)}$$

Change of Base Property for Logarithms. Let $a, b > 0$, w/ $a, b \neq 1$, and $M > 0$. Then $\log_a(M) = \frac{\log_b(M)}{\log_b(a)}$.

In particular, for our calculators, this means

$$\log_a(M) = \frac{\log(M)}{\log(a)} = \frac{\ln(M)}{\ln(a)}.$$

Ex $\log_7(82) = \frac{\ln(82)}{\ln(7)} \approx 2.26461$

Section 3.4

Suppose we want to solve for x in $3^{x+1} = 81$.
How do we do it?

Expressing Each Side As a Power of the Same Base

1. Rewrite equation in the form of $b^M = b^N$.
2. Set $M = N$.
3. Solve.

Ex $3^{x+1} = 81 = 3^4 \Rightarrow x+1 = 4 \Rightarrow x = 3$.

Use Logarithms to Solve Exponential Equations

1. Isolate exponential expression
2. Take a logarithm (w/ same base) of both sides.
3. Simplify w/ power rule for logarithms.
4. Solve.

Ex $3^{x+1} = 81 \Rightarrow \ln(3^{x+1}) = \ln(81)$

$$\Rightarrow (x+1)\ln(3) = \ln(81)$$

$$\Rightarrow x+1 = \frac{\ln(81)}{\ln(3)}$$

$$\Rightarrow x = \frac{\ln(81)}{\ln(3)} - 1 \quad \cdots\text{simplify}\cdots = 4 \cdot \frac{\ln(3)}{\ln(3)} - 1 = 3.$$

Use Definition of a Logarithm

1. Express as $\log_b(M) = c$.
2. Rewrite as $b^c = M$.
3. Solve.

Ex $\log_3(2x-6)=15 \Rightarrow 3^{15}=2x-6 \Rightarrow 2x=3^{15}+6 \Rightarrow x=\frac{1}{2}(3^{15})+3.$

Using 1-to-1 Property of Logs

1. Rewrite as $\log_b(M)=\log_b(N)$
2. Set $M=N$.
3. Solve.

Ex $\log(x)=\log(2)-\log(15) \Rightarrow \log(x)=\log\left(\frac{2}{15}\right) \Rightarrow x=\frac{2}{15}.$

Section 3.5

Def The function $f(t) = a_0 e^{kt}$ models **exponential growth** if $k > 0$, and **exponential decay** if $k < 0$. k

We have already seen examples of exponential growth (compound interest) and exponential decay (radioactive decay). Let's discuss a few more.

Ex (Population Growth) Populations tend to grow exponentially. The growth model $A = 4.1e^{0.01t}$ represents New Zealand's population t years after 2006. The growth rate is 0.01. When will the population double in size? At $t=0$, ie, in 2006, the population was 4.1 million. So we want to solve for t in the equation $8.2 = 4.1e^{0.01t}$.

$$8.2 = 4.1e^{0.01t} \Rightarrow 2 = e^{0.01t} \Rightarrow \ln(2) = 0.01t \Rightarrow t \approx 69.31.$$

So in the year 2075, New Zealand will double its population from 2006.

Ex Carbon dating of old objects is done by measuring the amount of carbon C-14 present in a sample, determining the amount of carbon C-14 that should have been in the sample, and then using the half-life of C-14 to figure out the sample's age. The accuracy lies in the fact that C-14's half life is about 5700 years. Its exponential decay function is $A(t) = A_0 e^{-0.000121t}$. Suppose a sample of clay from an ancient pot contains only 12% of its original C-14. Then

$$0.12A_0 = A_0 e^{-0.000121t} \Rightarrow 0.12 = e^{-0.000121t} \Rightarrow \ln(0.12) = -0.000121t,$$ so

solving for t, we see that the pot is $t \approx 16861$ year old.

LAST TIME WE TALKED ABOUT HALF-LIFE, OUR MODEL LOOKED SOMETHING LIKE $A = A_0\left(\frac{1}{2}\right)^t$, SO WHY DOES C-14 USE $A = A_0 e^{-0.000121t}$? IS IT SPECIAL? IN FACT, IT ISN'T. ANY EXPONENTIAL MODEL CAN BE WRITTEN AS $A = A_0 e^{kt}$.

REWRITING $y = ab^x$ AS AN EXPONENTIAL GROWTH/DECAY MODEL

SINCE $e^{\ln(x)} = x$, $e^{\ln(b^x)} = b^x$, HENCE $y = ab^x = ae^{\ln(b^x)} = ae^{x\ln(b)}$.

Ex TO CONVINCE US THAT THIS WORKS, LET'S FIGURE OUT THE EXPONENTIAL MODEL FOR CARBON C-14. IT HAS A HALF LIFE OF 5715 YEARS, SO

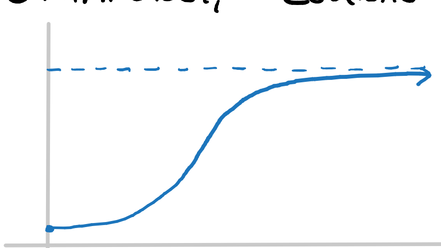
$$A = A_0\left(\frac{1}{2}\right)^{t/5715} = A_0 \exp\left(\frac{\ln(\frac{1}{2})t}{5715}\right) \approx A_0 e^{-0.000121t}.$$

EUREKA!

NOW, IT IS TRUE THAT NOTHING IN THE REAL WORLD CAN GROW EXPONENTIALLY FOREVER – EVENTUALLY THINGS LIKE FOOD SUPPLY WILL INHIBIT GROWTH OF A POPULATION. FOR THIS WE HAVE A DIFFERENT GROWTH MODEL.

DEF A LOGISTIC GROWTH MODEL IS GIVEN BY $A = \frac{c}{1 + ae^{-bt}}$, WHERE a, b, c ARE CONSTANTS AND $b, c > 0$. THE LIMITING SIZE IS THE HORIZONTAL ASYMPTOTE AS $t \to \infty$, WHICH IS $y = c$.

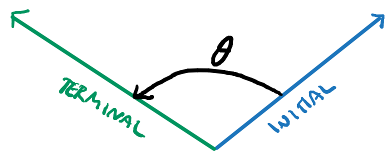
GRAPHICALLY, LOGISTIC MAPS LOOK LIKE



Ex THE WORLD'S POPULATION, IN BILLIONS, t YEARS AFTER 1949, IS GIVEN BY $f(t) = \dfrac{11.82}{1 + 3.81e^{-0.027t}}$, WHICH MEANS THAT SCIENTISTS EXPECT OUR EARTH IS INCAPABLE OF SUSTAINING MORE THAN 11.82 BILLION PEOPLE.

Section 4.1

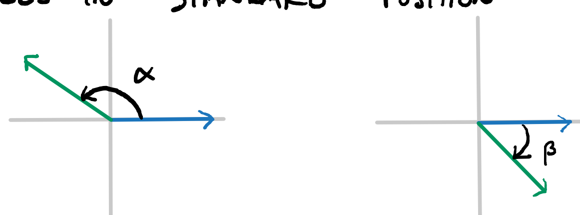
Def An **angle** is formed by two rays that share a common endpoint. One ray is called the **initial side** and the other is called the **terminal side**. The common endpoint is called the **vertex**.



An angle's direction and amount of rotation are from the initial side to the terminal side. Angles are usually labeled w/ Greek letters.

Def An angle is in **standard position** if the initial side is along the positive x-axis and the vertex is at the origin.

Ex Angles in standard position

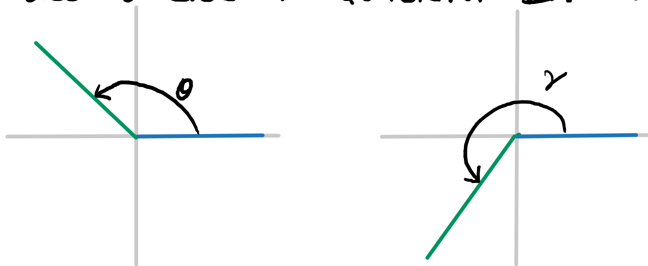


Def An angle is **positive** if the rotation is CCW, and **negative** if the rotation is CW.

Ex In the previous example, α is positive, β is negative.

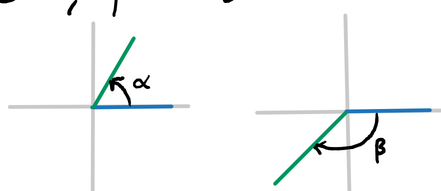
Def We say that an angle **lies in the quadrant** where its terminal side lies.

Ex Angle θ lies in Quadrant II. γ lies in Quadrant III



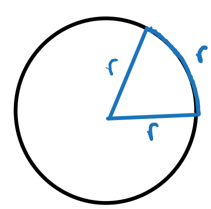
Def One **degree** (written 1°), is a measurement of the amount of rotation of an angle, and is $1/360^{th}$ of a complete rotation.

Ex $\alpha = 60^\circ$, $\beta = -135^\circ$



Def One **radian** (written just 1 or 1 rad) is the measure of the central angle of a circle that intercepts an arc equal in length to the radius of the circle.

In simpler terms, it's the measure of the angle that makes the three sides (two straight, one curved) in the picture to the right have the same length.



We can lay $2\pi \approx 6.28$ of these arcs along the circle, so this means that, in radians, a full rotation has measure 2π.

So, 2π rad $= 360°$ $\Rightarrow$ π rad $= 180°$. Dividing one side by the other leads us to the following:

Converting Between Radians and Degrees:

- Radians to Degrees: multiply the angle by $\frac{180°}{\pi}$.
- Degrees to Radians: multiply the angle by $\frac{\pi}{180°}$.

Ex $45° = 45°\left(\frac{\pi}{180°}\right) = \frac{\pi}{4}$ rad

$198° = 198°\left(\frac{\pi}{180°}\right) = \frac{11\pi}{10}$

$-\frac{\pi}{6} = -\frac{\pi}{6}\left(\frac{180°}{\pi}\right) = -30°$

$1.71\pi = 1.71\pi\left(\frac{180°}{\pi}\right) = 307.8°$

Def Two angles are **coterminal** if they differ by a multiple of $360°$ or 2π rad.

Ex $180°$ and $-180°$ are coterminal: $180° = -180° + 360°$.

$\frac{25\pi}{6}$ and $\frac{\pi}{6}$ are coterminal: $\frac{25\pi}{6} = \frac{\pi}{6} + 2(2\pi) = \frac{\pi}{6} + \frac{24\pi}{6}$

Section 4.2

Def The **unit circle** is the circle of radius 1, centered at the origin.

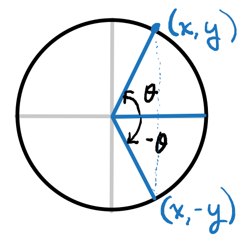
Def If θ is an angle and $P(x,y)$ is the corresponding point on the unit circle, we have the following functions:

$\sin(\theta) = y$; $\cos(\theta) = x$; $\tan(\theta) = \frac{y}{x}$, $x \neq 0$;

$\csc(\theta) = \frac{1}{y}$, $y \neq 0$; $\sec(\theta) = \frac{1}{x}$, $x \neq 0$; $\cot(\theta) = \frac{x}{y}$, $y \neq 0$.

SINCE θ CAN BE ANY REAL NUMBER, THE DOMAIN FOR SINE AND COSINE IS $(-\infty, \infty)$. SINCE $x = \cos\theta$ AND $y = \sin\theta$, THE RANGE FOR COSINE AND SINE IS $[-1, 1]$.

NOTICE THAT ON THE UNIT CIRCLE, WHEN WE PLUG IN θ AND $-\theta$, WE HAVE THAT THE x-VALUES STAY THE SAME, BUT THE y-VALUES CHANGE SIGN. SO IN TERMS OF OUR TRIG FUNCTIONS:



$$\sin(-\theta) = -\sin(\theta) \qquad \csc(-\theta) = -\csc(\theta)$$
$$\cos(-\theta) = \cos(\theta) \qquad \sec(-\theta) = \sec(\theta)$$
$$\tan(-\theta) = -\tan(\theta) \qquad \cot(-\theta) = -\tan(\theta)$$

THIS MEANS THAT COS AND SEC ARE <u>EVEN</u>; AND sin, csc, tan, cot ARE <u>ODD</u>.

RELATIONSHIPS BETWEEN TRIG FUNCTIONS: EACH OF THESE FOLLOWS FROM THE DEFINITIONS.

$$\sin(\theta) = \frac{1}{\csc(\theta)} \qquad \csc(\theta) = \frac{1}{\sin(\theta)}$$
$$\cos(\theta) = \frac{1}{\sec(\theta)} \qquad \sec(\theta) = \frac{1}{\cos(\theta)}$$
$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{1}{\cot(\theta)} \qquad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} = \frac{1}{\tan(\theta)}$$

THIS MEANS THAT, GIVEN SOME VALUE FOR ANY TWO TRIG FUNCTIONS, WE CAN DETERMINE THE VALUES FOR THE OTHER FOUR.

Section 4.2 (Continued)

Def A function is called **periodic** if there exists a real number $p > 0$ s.t. $f(x+p) = f(x)$ for all x in the domain of f. The number p is called the **period** of f.

Periodic Properties of Sine, Cosine, Cosecant, and Secant

$\sin(\theta + 2\pi) = \sin(\theta)$ and $\cos(\theta + 2\pi) = \cos(\theta)$

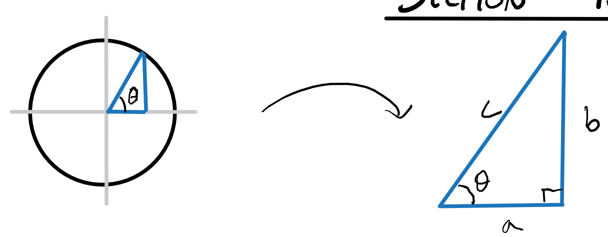
$\csc(\theta + 2\pi) = \csc(\theta)$ and $\sec(\theta + 2\pi) = \sec(\theta)$

Each of these has period 2π.

Periodic Properties of Tangent and Cotangent

$\tan(\theta + \pi) = \tan(\theta)$ and $\cot(\theta + \pi) = \cot(\theta)$

Section 4.3



By merely drawing a triangle, we get

Def Each trig function can be defined in terms of the right triangle above:

$\sin(\theta) = \frac{b}{c}$ $\qquad \csc(\theta) = \frac{c}{b}$

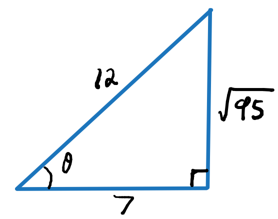
$\cos(\theta) = \frac{a}{c}$ $\qquad \sec(\theta) = \frac{c}{a}$

$\tan(\theta) = \frac{b}{a}$ $\qquad \cot(\theta) = \frac{a}{b}$

Notice that, when $c = 1$, these exactly align with our unit circle definitions.

Knowing only one trig function value, we can determine the others.

Suppose $\sec(\theta) = \frac{12}{7}$. Then $7^2 + b^2 = 12^2$ $\Rightarrow b^2 = 144 - 49 \Rightarrow b = \sqrt{95}$ is the length of our missing side. So

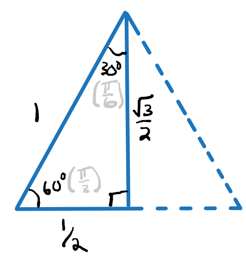


$\sin(\theta) = \frac{\sqrt{95}}{12}$ $\qquad \csc(\theta) = \frac{12}{\sqrt{95}}$

$\cos(\theta) = \frac{7}{12}$ $\qquad \cot(\theta) = \frac{7}{\sqrt{95}}$

$\tan(\theta) = \frac{\sqrt{95}}{7}$

An important angle is $60° = \frac{\pi}{3}$. If you recall, a triangles internal angles sum to $180° = \pi$, and in an equilateral triangle, each angle is $\frac{\pi}{3}$. So, consider the equilateral triangle w/ side length 1, and examine the right triangle formed by cutting it in half. With the Pythagorean theorem, the remaining side length is $\frac{\sqrt{3}}{2}$. Thus



$\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$ $\qquad \csc(\frac{\pi}{3}) = \frac{2\sqrt{3}}{3}$ $\qquad \sin(\frac{\pi}{6}) = \frac{1}{2}$ $\qquad \csc(\frac{\pi}{6}) = 2$

$\cos(\frac{\pi}{3}) = \frac{1}{2}$ $\qquad \sec(\frac{\pi}{3}) = 2$ $\qquad \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$ $\qquad \sec(\frac{\pi}{6}) = \frac{2\sqrt{3}}{3}$

$\tan(\frac{\pi}{3}) = \sqrt{3}$ $\qquad \cot(\frac{\pi}{3}) = \frac{\sqrt{3}}{3}$ $\qquad \tan(\frac{\pi}{6}) = \frac{\sqrt{3}}{3}$ $\qquad \cot(\frac{\pi}{6}) = \sqrt{3}$

Def Two angles are **complements** if they sum to $90°$. Two functions are called **cofunctions** if $f(\theta) = g(90° - \theta)$.

Ex $30° = \frac{\pi}{6}$ and $60° = \frac{\pi}{3}$ are complements.

sin and cos are cofunctions.

Remark: "cosine" is short for "complement's sine."

COFUNCTION IDENTITIES (IF θ IS IN RADIANS, REPLACE $90°$ w/ $\frac{\pi}{2}$)

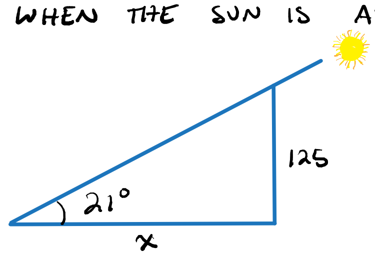
$\sin(\theta) = \cos(90° - \theta)$ $\qquad$ $\csc(\theta) = \sec(90° - \theta)$

$\cos(\theta) = \sin(90° - \theta)$ $\qquad$ $\sec(\theta) = \csc(90° - \theta)$

$\tan(\theta) = \cot(90° - \theta)$ $\qquad$ $\cot(\theta) = \tan(90° - \theta)$

Ex SINCE SINE AND COSINE ARE COFUNCTIONS, $\cos(38°)$ AND $\sin(52°)$ HAVE THE SAME VALUE. ALSO, $\sec\left(\frac{\pi}{12}\right) = \csc\left(\frac{5\pi}{12}\right)$.

Ex FIND THE LENGTH OF THE SHADOW CAST BY A 125 ft TOWER WHEN THE SUN IS AT $21°$ ABOVE THE HORIZON.



$$\tan\left(\frac{\pi}{8}\right) = \frac{125 \text{ ft}}{x}$$

$$\Rightarrow \quad x = \frac{125 \text{ ft}}{\tan(21°)} \approx 325.64 \text{ ft}.$$

Section 4.4

Let θ be any angle in standard position, and $P(x,y)$ a point on the terminal side of θ. If $r=\sqrt{x^2+y^2}$ is the distance from $(0,0)$ to (x,y), the six trig functions are given by

$$\sin(\theta)=\frac{y}{r} \qquad \csc(\theta)=\frac{r}{y},\ y\neq 0$$
$$\cos(\theta)=\frac{x}{r} \qquad \sec(\theta)=\frac{r}{x},\ x\neq 0$$
$$\tan(\theta)=\frac{x}{y},\ y\neq 0 \qquad \cot(\theta)=\frac{y}{x},\ x\neq 0$$

Ex Let $P(-7,1)$ be on the terminal side of θ. Then

$$r=\sqrt{(-7)^2+1^2}=\sqrt{50}=5\sqrt{2}$$
$$\sin(\theta)=\frac{1}{5\sqrt{2}}=\frac{\sqrt{2}}{10} \qquad \csc(\theta)=5\sqrt{2}$$
$$\cos(\theta)=-\frac{7}{5\sqrt{2}}=\frac{-7\sqrt{2}}{10} \qquad \sec(\theta)=-\frac{5\sqrt{2}}{7}$$
$$\tan(\theta)=-\frac{1}{7} \qquad \tan(\theta)=-7$$

One thing we notice is that the sign of the trig functions depend on the quadrant.

QII	QI
$\sin(\theta)>0$	$\sin(\theta)>0$
$\cos(\theta)<0$	$\cos(\theta)>0$
QIII	**QIV**
$\sin(\theta)<0$	$\sin(\theta)<0$
$\cos(\theta)<0$	$\cos(\theta)>0$

So in fact, we only need one trig function and info about the quadrant to determine the remaining trig functions.

Ex $\tan(\theta)=-\frac{1}{3}$, $\sin(\theta)>0$.

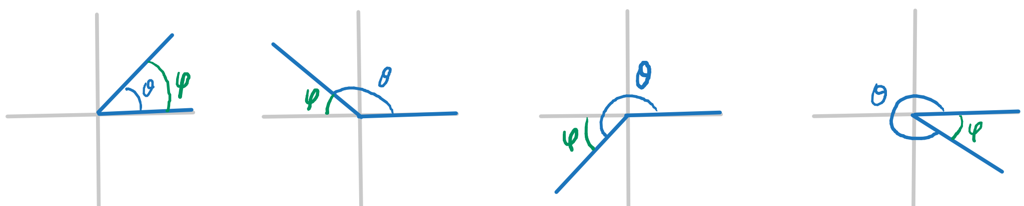
Since $\sin(\theta)>0$ and $\tan(\theta)<0$, $\cos(\theta)<0$.

$r=\sqrt{1^2+(-3)^2}=\sqrt{10}$, so

$$\sin(\theta)=\frac{1}{\sqrt{10}}=\frac{\sqrt{10}}{10} \qquad \csc(\theta)=\sqrt{10} \qquad \cot=-3.$$
$$\cos(\theta)=-\frac{3}{\sqrt{10}}=-\frac{3\sqrt{10}}{10} \qquad \sec(\theta)=-\frac{\sqrt{10}}{3}$$

Def Let θ be a nonacute (ie, $|\theta| > 90°$ or $\frac{\pi}{2}$) angle. The **reference angle** is the angle φ formed by the terminal side of θ and the x-axis.

Ex



Ex $\theta = 31°$, $\varphi = 31°$

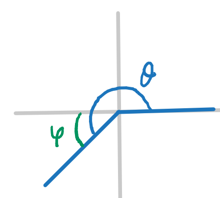
$\theta = 170°$, $\varphi = 10°$

$\theta = -170°$, $\varphi = 10°$

$\theta = \frac{13\pi}{6}$, $\varphi = \frac{\pi}{6}$

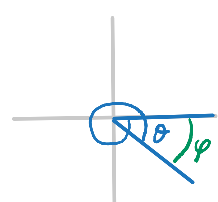
Using Reference Angles to Find Trig Values

Ex $\theta = \frac{5\pi}{4}$, so $\varphi = \frac{\pi}{4}$. Now, $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, and since θ is in QIII, $\sin(\theta) < 0$. Hence $\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$.



The whole point of these reference angles is that it allows us to think about only one quadrant of the unit circle, and know that every other quadrant results in trig values that are the same up to a sign.

Ex $\theta = -\frac{7\pi}{3}$, so $\varphi = \frac{\pi}{3}$. Now, $\sec\left(\frac{\pi}{3}\right) = \frac{1}{\cos\left(\frac{\pi}{3}\right)} = \frac{1}{1/2} = 2$ and θ is in QIV, so $\cos\theta > 0$ $\Rightarrow \sec\theta > 0$. Hence $\sec\left(-\frac{7\pi}{3}\right) = 2$.

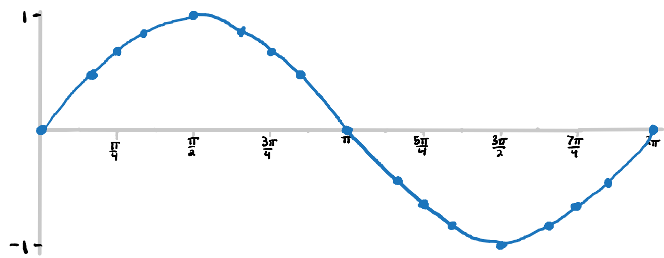


Section 4.5

Graphing $y=\sin(x)$

x	$\sin(x)$
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$\frac{1}{2}$

x	$\sin(x)$
π	0
$\frac{7\pi}{6}$	$-\frac{1}{2}$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$\frac{3\pi}{2}$	-1
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$
$\frac{11\pi}{6}$	$-\frac{1}{2}$



Graph of $y=\sin(x)$

Above we see, graphically, what sine looks like. Recall that sine had a period 2π. What this means is, if we continued graphing this, we would see the same shape repeated over and over.

One thing we notice is that the graph of $\sin(x)$ reaches its maximum at $\frac{1}{4}$ of the period, it's minimum at $\frac{3}{4}$ of the period, and has x-intercepts at the beginning, middle and end of the period.

Let's look at some transformations of $y=\sin(x)$.

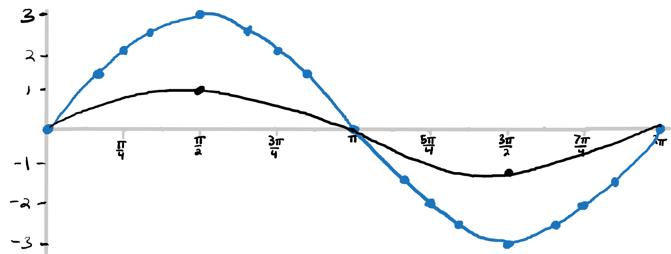
Amplitude

Def Let $y=A\sin(x)$ for $A\neq 0$. Then $|A|$ is called the **amplitude** of $y=A\sin(x)$. This function has range $[-|A|,|A|]$.

Ex $y = 3\sin(x)$ IS A VERTICAL STRETCH OF $\sin(x)$, AS EXPECTED.

x	$3\sin(x)$
0	0
$\frac{\pi}{6}$	$\frac{3}{2}$
$\frac{\pi}{4}$	$\frac{3\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{3\sqrt{3}}{2}$
$\frac{\pi}{2}$	3
$\frac{2\pi}{3}$	$\frac{3\sqrt{3}}{2}$
$\frac{3\pi}{4}$	$\frac{3\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$\frac{3}{2}$

x	$\sin(x)$
π	0
$\frac{7\pi}{6}$	$-\frac{3}{2}$
$\frac{5\pi}{4}$	$-\frac{3\sqrt{3}}{2}$
$\frac{4\pi}{3}$	$-\frac{3\sqrt{3}}{2}$
$\frac{3\pi}{2}$	-3
$\frac{5\pi}{3}$	$-\frac{3\sqrt{3}}{2}$
$\frac{7\pi}{4}$	$-\frac{3\sqrt{2}}{2}$
$\frac{11\pi}{6}$	$-\frac{3}{2}$



GRAPH OF $y = \sin(x)$

GRAPH OF $y = 3\sin(x)$

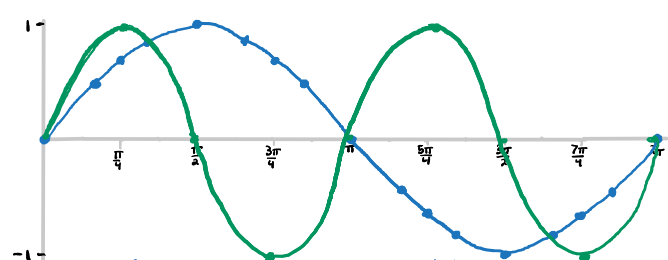
NOTICE THAT $y = A\sin(x)$ AND $y = \sin(x)$ HAVE THE SAME PERIOD, AND SAME LOCATIONS OF MAXIMA, MINIMA, AND X-INTERCEPTS.

PERIOD

Ex $y = \sin(2x)$ IS A HORIZONTAL SHRINK, AS EXPECTED.

x	$\sin(2x)$
0	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$\frac{3\pi}{4}$	-1
$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$

x	$\sin(2x)$
π	0
$\frac{7\pi}{6}$	$\frac{\sqrt{3}}{2}$
$\frac{5\pi}{4}$	1
$\frac{4\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{3\pi}{2}$	0
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$\frac{7\pi}{4}$	-1
$\frac{11\pi}{6}$	$-\frac{\sqrt{3}}{2}$



GRAPH OF $y = \sin(x)$

GRAPH OF $y = \sin(2x)$

$y = A\sin(Bx)$ HAS PERIOD $\frac{2\pi}{B}$.

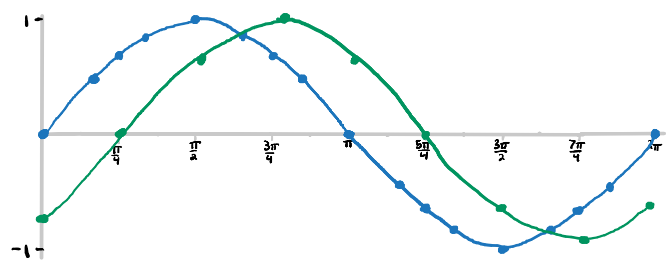
Section 4.5 (Continued)

Last time we saw how vertical and horizontal shrinks/stretches affected the graph of $y=\sin(x)$. What about horizontal shifts?

Ex $y=\sin\left(x-\frac{\pi}{4}\right)$ is a horizontal shift right by $\frac{\pi}{2}$, as expected

x	$\sin\left(x-\frac{\pi}{4}\right)$
0	$-\frac{\sqrt{2}}{2}$
$\frac{\pi}{6}$	-0.259
$\frac{\pi}{4}$	0
$\frac{\pi}{3}$	0.259
$\frac{\pi}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{2\pi}{3}$	0.966
$\frac{3\pi}{4}$	1
$\frac{5\pi}{6}$	0.966

x	$\sin\left(x-\frac{\pi}{4}\right)$
π	$\frac{\sqrt{2}}{2}$
$\frac{7\pi}{6}$	0.259
$\frac{5\pi}{4}$	0
$\frac{4\pi}{3}$	-0.259
$\frac{3\pi}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{5\pi}{3}$	-0.966
$\frac{7\pi}{4}$	-1
$\frac{11\pi}{6}$	-0.966



Graph of $y=\sin(x)$

Graph of $y=\sin\left(x-\frac{\pi}{4}\right)$

More generally, if $y=A\sin(Bx-C)=A\sin\left(B\left(x-\frac{C}{B}\right)\right)$, $A,B,C\neq 0$, the graph has amplitude A, period $\frac{2\pi}{B}$, and phase shift $\frac{C}{B}$.

Ex Determine the amplitude, period, and phase shift of $y=3\sin\left(2x-\frac{\pi}{2}\right)$. Then graph one period of the function.

Amplitude: $|3|=3$

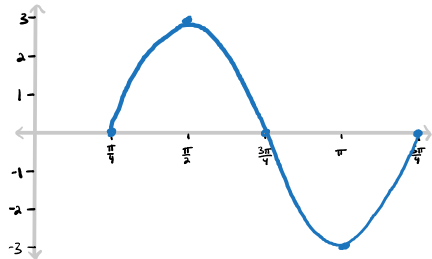
Period: $\frac{2\pi}{2}=\pi$

Phase shift: $\frac{\pi/2}{2}=\frac{\pi}{4}$

Recall that the x-intercepts lie at the end points of the period (so $x=\frac{\pi}{4}$, $x=\frac{\pi}{4}+\pi=\frac{5\pi}{4}$), and at the midpoint $\left(x=\frac{\pi}{4}+\frac{\pi}{2}=\frac{3\pi}{4}\right)$.

Since the graph is not reflected, the max is halfway between the first two intercepts (so $x=\frac{\pi}{4}+\frac{\pi}{4}=\frac{\pi}{2}$) and the max is halfway between the second two (so $x=\frac{\pi}{4}+\frac{3\pi}{4}=\pi$).

Ex (Cont'd) So, the graph of one period of $y = 3\sin\left(2x - \frac{\pi}{2}\right)$ is

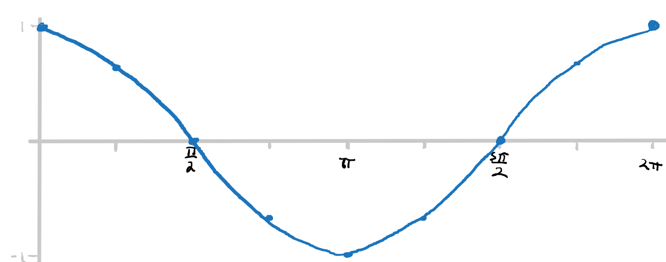


Procedure Recap:

1) Identify amplitude A, period $\frac{2\pi}{B}$, and phase shift $\frac{C}{B}$.
2) Identify x-intercepts, max, and min (these occur every $\frac{C}{B} + n\frac{\text{period}}{4}$, $n = 0, \ldots, 4$).
3) For x-values above, plot the 5 points.
4) Connect w/ a smooth curve.

Graphing $y = \cos(x)$.

x	$\cos(x)$
0	1
$\frac{\pi}{2}$	0
π	-1
$\frac{3\pi}{2}$	0
2π	1



We notice a few things: the x-intercepts, max, and min are all still at $\frac{\text{period}}{4}$ intervals, and the graph of $y = \cos(x)$ is the same as the graph of $y = \sin\left(x - \frac{\pi}{2}\right)$ (the latter is not surprising as sine and cosine are cofunctions).

NOTE: FOR $y = A\cos(Bx - C)$, $A, B, C \neq 0$, THE AMPLITUDE, PERIOD, AND PHASE SHIFT ARE ALL STILL DEFINED IN THE SAME WAY.

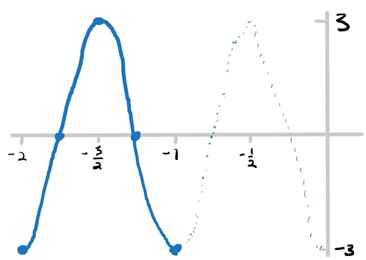
Ex $y = -3x(2\pi x + 4\pi)$ NOTE THAT THE NEGATIVE LEADING TERM IS A REFLECTION ABOUT THE X-AXIS.

AMPLITUDE: $|-3| = 3$

PERIOD: $\frac{2\pi}{2\pi} = 1$

PHASE SHIFT: $\frac{-4\pi}{2\pi} = -2$

x	-2	$-2+\frac{1}{4}$	$-2+\frac{2}{4}$	$-2+\frac{3}{4}$	$-2+\frac{4}{4}$
$-3\cos(2\pi x + 4\pi)$	-3	0	3	0	-3



THE PROCEDURE FOR GRAPHING BOTH $y = A\sin(Bx - C)$ AND $y = A\cos(Bx - C)$.

REMARK. ALL TRANSFORMATIONS OF $\sin(x)$ AND $\cos(x)$ BEHAVE AS YOU'D EXPECT, WE JUST DON'T HAVE ANY COOL NAMES FOR VERTICAL SHIFTS.

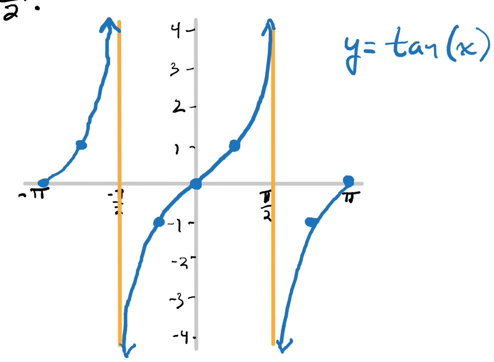
Ex $y = 3\sin\left(\frac{2\pi}{365}(x - 79)\right) + 12$ REPRESENTS THE NUMBER OF HOURS OF DAYLIGHT IN BOSTON, X-DAYS AFTER JANUARY 1.

THE AMPLITUDE IS $|3| = 3$, THE PERIOD IS $\frac{2\pi}{(2\pi/365)}$, AND THE PHASE SHIFT IS 79. THE LONGEST DAY OF THE YEAR OCCURS AT $x = 79 + \frac{365}{4}$ days, WHICH IS JUNE 20th. THE AMOUNT OF DAYLIGHT IS 15 HOURS. THE SHORTEST DAY OF THE YEAR IS AT $x = 79 + \frac{3(365)}{4}$ days, WHICH IS DECEMBER 19. THE AMOUNT OF DAYLIGHT IS ONLY 9 HOURS.

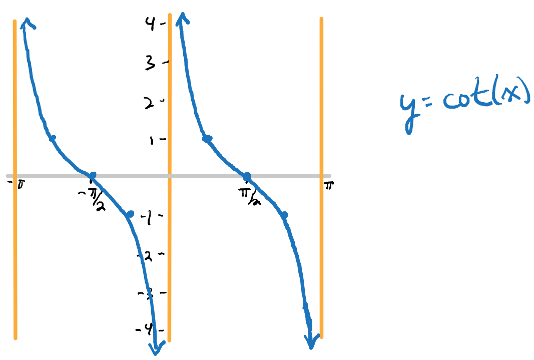
Section 4.6

This will not be on the exam, but I think it's good to see at some point.

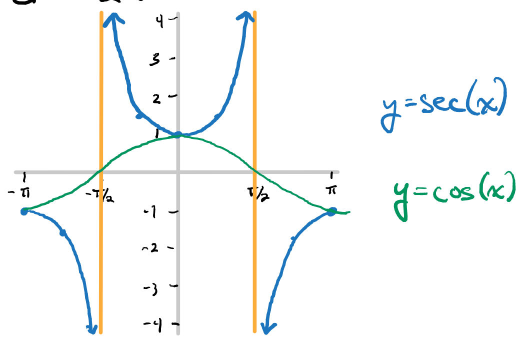
Since $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$, the domain is
$\{\theta \mid \cos(\theta) \neq 0\} = \{\theta \mid \theta \neq \frac{2k+1}{2}\pi$, k an integer$\}$. Indeed, tangent also has vertical asymptotes at each odd multiple of $\frac{\pi}{2}$.



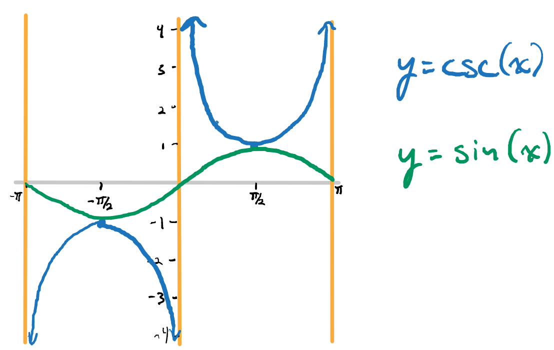
Since $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$, the domain is
$\{\theta \mid \sin(\theta) \neq 0\} = \{\theta \mid \theta \neq k\pi$, k an integer$\}$. Indeed, $y = \cot(x)$ has vertical asymptotes at each multiple of π.



SINCE $\sec(\theta) = \frac{1}{\cos(\theta)}$, $y = \sec(x)$ HAS THE SAME DOMAIN AS $y = \tan(x)$. INDEED, IT HAS VERTICAL ASYMPTOTES AT EACH ODD MULTIPLE OF $\frac{\pi}{2}$.



SINCE $\csc(\theta) = \frac{1}{\sin(\theta)}$, $y = \csc(x)$ HAS THE SAME DOMAIN AS $y = \cot(x)$. INDEED, $y = \csc(x)$ HAS VERTICAL ASYMPTOTES AT MULTIPLES OF π.



Section 4.8

On the domain $(-\infty, \infty)$, we see that $f(x) = \sin(x)$ is very much not one-to-one, and thus not invertible. However, restricting the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$, it is very much invertible.

Def The inverse sine function, denoted $\sin^{-1}$ or arcsin, is the inverse of $y = \sin(x)$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Thus $y = \sin^{-1} x$ is equivalent to $x = \sin(y)$.

$y = \sin^{-1}(x) \neq \frac{1}{\sin(x)}$. The -1 exponent is meant to denote the inverse function, like $f^{-1}(x)$. Of course, this conflicts with notation like $\sin^2(x) = (\sin(x))^2$. For this reason, I will use $y = \arcsin(x)$ whenever possible to avoid conflicting notation.

[Graph]

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Ex $\sin^{-1}\left(\frac{1}{2}\right) = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$\cos\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) = \cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

Def The inverse cosine function, denoted $\cos^{-1}$ or arccos, is the inverse of $y = \cos(x)$, $0 \le x \le \pi$. $y = \cos^{-1}(x)$ is equivalent to $x = \cos(y)$.

[Graph]

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

Ex

Def THE INVERSE TANGENT FUNCTION, DENOTED $\tan^{-1}$ OR $\arctan$, IS THE INVERSE OF $y = \tan(x)$, $\frac{\pi}{2} < x < \frac{\pi}{2}$. $y = \tan^{-1}(x)$ IS EQUIVALENT TO $\tan(y) = x$

[GRAPH]

θ	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\tan\theta$	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Ex

INVERSE PROPERTIES:

$\sin(\sin^{-1}(x)) = x$, FOR x IN $[-1, 1]$

$\sin^{-1}(\sin(x)) = x$, FOR x IN $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$\cos(\cos^{-1}(x)) = x$, FOR x IN $[-1, 1]$

$\cos^{-1}(\cos(x)) = x$, FOR x IN $[0, \pi]$

$\tan(\tan^{-1}(x)) = x$, FOR x IN $(-\infty, \infty)$

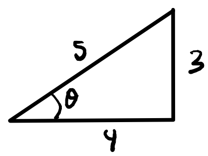
$\tan^{-1}(\tan(x)) = x$, FOR x IN $(-\frac{\pi}{2}, \frac{\pi}{2})$

Ex $\sin^{-1}(\sin(0)) = 0$ BECAUSE 0 IS IN $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

$\sin^{-1}(\sin(\pi)) \neq \pi$, BECAUSE π IS NOT IN $[-\frac{\pi}{2}, \frac{\pi}{2}]$. INSTEAD, WE HAVE TO ACTUALLY EVALUATE: $\sin^{-1}(\sin(\pi)) = \sin^{-1}(0) = 0$.

Ex FIND THE EXACT VALUE OF $\sin(\tan^{-1}(\frac{3}{4}))$. $\frac{3}{4}$ IS IN DOMAIN.

$\tan^{-1}(\frac{3}{4}) = \theta \Leftrightarrow \tan(\theta) = \frac{3}{4}$, WHERE $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, SO $\theta > 0$.



$r = \sqrt{3^2 + 4^2} = \sqrt{25}$. THEN

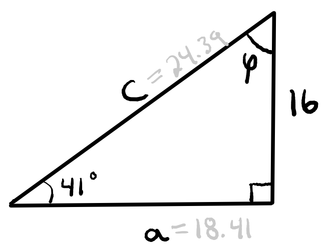
$\sin(\tan^{-1}(\frac{3}{4})) = \sin(\theta) = \frac{3}{r} = \frac{3}{5}$

Section 4.8

Solving Right Triangles:

Given any two side lengths OR a side length and an angle in a right triangle, we can solve for the remaining information.

Ex



Find a, c, φ.

Notice that $\tan(41°) = \frac{16}{a}$, so $a = \frac{16}{\tan(41°)} \approx 18.41$

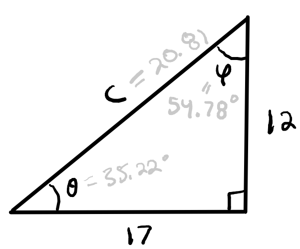
Now that we have a, we can solve for c in a few different ways:

$\cos(41°) = \frac{18.41}{c}$, $c = \sqrt{18.41^2 + 16^2}$. In either case, we get $c \approx 24.39$.

To find φ, we have even more options:

$\sin(\varphi) = \frac{18.41}{24.39}$, $\cos(\varphi) = \frac{16}{24.39}$, $\tan(\varphi) = \frac{18.41}{16}$, or even avoiding inverse trig and remembering $41° + \varphi = 90°$. We get $\varphi = 49°$.

Ex



Find c, θ, φ.

By the Pythagorean theorem, $c = \sqrt{12^2 + 17^2} \approx 20.81$

Now, $\tan(\theta) = \frac{12}{17}$, or $\cot(\theta) = \frac{17}{12}$, or $\sin(\theta) = \frac{12}{20.81}$, or $\cos(\theta) = \frac{17}{20.81}$. Using $\theta = \arctan\left(\frac{12}{17}\right) \approx 35.22°$.

Similarly, there are a million-and-five ways to get φ. Choosing $\sin(\varphi) = \frac{17}{20.81}$, we have $\varphi = \arcsin\left(\frac{17}{20.81}\right) \approx 54.78°$

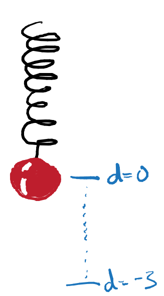
THE IMPORTANT THING WHEN SOLVING A TRIANGLE IS THAT YOU NEED AT LEAST ONE SIDE LENGTH. (IF YOU RECALL FROM GEOMETRY, THERE ARE MANY TRIANGLES w/ SAME INTERNAL ANGLES, BUT DIFFERENT SIDE LENGTHS, HENCE AAA SIMILARITY AND NOT AAA CONGRUENCE.)

ONE IMPORTANT APPLICATION OF TRIG COMES IN THE STUDY OF SIMPLE HARMONIC MOTION IN PHYSICS.

THE CANONICAL EXAMPLE OF SIMPLE HARMONIC MOTION IS A MASS ON A SPRING. (see animations.physics.unsw.edu.au/jw/SHM.htm). THE MASS HAS AN EQUILIBRIUM POSITION WITH THE MASS ATTACHED. IF WE PULL THE MASS AND STRETCH THE SPRING. IGNORING THINGS LIKE FRICTION AND AIR RESISTANCE, THE MASS WOULD STAY BOBBING UP AND DOWN FOREVER.

DEF AN OBJECT THAT MOVES ON A COORDINATE AXIS IS IN **SIMPLE HARMONIC MOTION** IF ITS DISTANCE, d, FROM THE EQUILIBRIUM POSITION AT TIME t IS MODELED BY $d = a\cos(\omega t)$ OR $d = a\sin(\omega t)$. THIS MOTION HAS **AMPLITUDE** $|a|$. THE **PERIOD** IS GIVEN BY $\frac{2\pi}{\omega}$, WHERE $\omega > 0$.

EX



SUPPOSE A BALL IS ATTACHED TO A SPRING AND PULLED DOWN 3 FEET FROM ITS EQUILIBRIUM POSITION. WHEN RELEASED, IT TAKES 5 SECONDS TO BOB UP AND COME BACK DOWN TO THE RELEASE POINT. THE PERIOD IS $\frac{2\pi}{\omega} = 5 \Rightarrow \omega = \frac{2\pi}{5}$. SINCE THE BALL STARTED DOWN, WE HAVE THAT $a = -3$. SO THE EQ. MODELING THE MOTION OF THE BALL IS $d = -3\cos\left(\frac{2\pi}{5}t\right)$.

The frequency of the ball's motion was $\frac{1}{5}$th of a cycle per second.

Def An object in simple harmonic motion given by $d = a\sin(\omega t)$ or $d = a\cos(\omega t)$ has frequency $f = \frac{\omega}{2\pi} = \frac{1}{\text{period}}$, $\omega > 0$.

Ex An object is in simple harmonic motion given by $d = 14\cos\left(\frac{\pi}{4}t\right)$. The frequency is $f = \frac{\omega}{2\pi} = \frac{\pi/4}{2\pi} = \frac{1}{8}$, so the period is 8 seconds.

Section 5.1

Fundamental Trig Identities

- Reciprocal Identities

$\sin\theta = \frac{1}{\csc\theta}$ $\qquad \cos\theta = \frac{1}{\sec\theta}$ $\qquad \tan\theta = \frac{1}{\cot\theta}$

$\csc\theta = \frac{1}{\sin\theta}$ $\qquad \sec\theta = \frac{1}{\cos\theta}$ $\qquad \cot\theta = \frac{1}{\tan\theta}$

- Quotient Identities

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$

- Pythagorean Identities

$\cos^2\theta + \sin^2\theta = 1$ $\qquad \tan^2\theta + 1 = \sec^2\theta$ $\qquad \cot^2\theta + 1 = \csc^2\theta$

- Even / Odd Identities

$\sin(-\theta) = -\sin(\theta)$ $\qquad \cos(-\theta) = \cos(\theta)$ $\qquad \tan(-\theta) = -\tan(\theta)$

$\csc(-\theta) = -\csc(\theta)$ $\qquad \sec(-\theta) = \sec(\theta)$ $\qquad \cot(-\theta) = -\cot(\theta)$

Techniques for Verifying Trig Identities:

- Work with each side independently
- Analyze the identity and look for ways to apply fundamental identities.
- Rewrite in terms of sines & cosines
- Factor out the greatest common factor
- Separate or combine fractions, $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$.
- Rewrite fractions w/ least common denominator
- Don't be afraid to stop and start over.

Ex $\quad \cos\theta \tan\theta \csc\theta = \left(\frac{\cancel{\cos\theta}}{1}\right)\left(\frac{\cancel{\sin\theta}}{\cancel{\cos\theta}}\right)\left(\frac{1}{\cancel{\sin\theta}}\right)$ $\qquad$ (Quotient / Reciprocal Idents)

$= 1$

Ex $\quad \cos^2\theta - \sin^2\theta = \cos^2\theta - (1 - \cos^2\theta)$ $\qquad$ (Pythag $\sin^2 + \cos^2 = 1$)

$= \cos^2\theta - 1 + \cos^2\theta$

$= 2\cos^2\theta - 1$

Ex $\csc\theta - \sin\theta = \frac{1}{\sin\theta} - \sin\theta$

$= \frac{1}{\sin\theta} - \frac{\sin^2\theta}{\sin\theta}$

$= \frac{1-\sin^2\theta}{\sin\theta}$

$= \frac{\cos^2\theta}{\sin\theta}$ (Pythagorean identity)

$= \left(\frac{\cos\theta}{\sin\theta}\right)\cos\theta$

$= \cot\theta \cos\theta$ (Quotient identity)

Ex $\frac{1+\cos t}{1-\cos t} = \frac{1+\cos t}{1-\cos t}\left(\frac{1+\cos t}{1+\cos t}\right)$

$= \frac{(1+\cos t)^2}{1-\cos^2 t}$

$= \frac{(1+\cos t)^2}{\sin^2 t}$ (Pythagorean Identity)

$= \left(\frac{1+\cos t}{\sin t}\right)^2$

$= \left(\frac{1}{\sin t} + \frac{\cos t}{\sin t}\right)^2$

$= (\csc t + \cot t)^2$ (Reciprocal / Quotient identities)

Ex $(\cos\theta - \sin\theta)^2 + (\cos\theta + \sin\theta)^2 = \cos^2\theta - 2\cos\theta\sin\theta + \sin^2\theta + \cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta$

$= 2\cos^2\theta + 2\sin^2\theta$

$= 2(\cos^2\theta + \sin^2\theta)$

$= 2$ (Pythagorean ident)

Ex $\frac{\cos^2 x - \sin^2 x}{1-\tan^2 x} = \frac{\cos^2 x - \sin^2 x}{1 - \sin^2 x/\cos^2 x}$ (Quotient ident)

$= \frac{\cos^2 x}{\cos^2 x}\left(\frac{\cos^2 x - \sin^2 x}{1 - \sin^2 x/\cos^2 x}\right)$

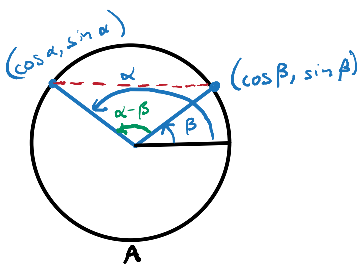
$= \cos^2 x\left(\frac{\cos^2 x - \sin^2 x}{\cos^2 x - \sin^2 x}\right)$

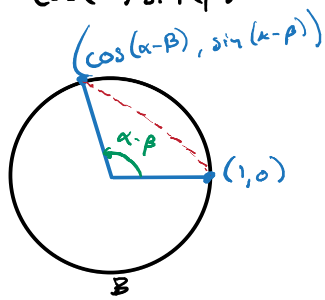
$= \cos^2 x$

Section 5.2

Theorem (Sum/Difference Formulas) For angles α, β,

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$
$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$





Proof By the distance formula, the length of the red dashed line in figure A is

$$d = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2}$$
$$= \sqrt{\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta + \sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta}$$
$$= \sqrt{(\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)}$$
$$= \sqrt{2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)}.$$

The length of the red dashed line in figure B is

$$d = \sqrt{(\cos(\alpha-\beta) - 1)^2 + (\sin(\alpha-\beta) - 0)^2}$$
$$= \sqrt{\cos^2(\alpha-\beta) - 2\cos(\alpha-\beta) + 1 + \sin^2(\alpha-\beta)}$$
$$= \sqrt{\cos^2(\alpha-\beta) + \sin^2(\alpha-\beta) + 1 - 2\cos(\alpha-\beta)}$$
$$= \sqrt{2 - 2\cos(\alpha-\beta)}$$

So we must have

$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta).$$

It follows then that

$$\cos(\alpha+\beta) = \cos(\alpha-(-\beta)) = \cos(\alpha)\cos(-\beta) + \sin(\alpha)\sin(-\beta)$$
$$= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

Recall now that $\cos(\frac{\pi}{2} - \theta) = \sin(\theta)$. So,

$$\sin(\alpha+\beta) = \cos\left(\tfrac{\pi}{2}-(\alpha+\beta)\right)$$
$$= \cos\left(\left(\tfrac{\pi}{2}-\alpha\right)-\beta\right)$$
$$= \cos\left(\tfrac{\pi}{2}-\alpha\right)\cos(\beta) + \sin\left(\tfrac{\pi}{2}-\alpha\right)\sin(\beta)$$
$$= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

AND IT FOLLOWS THAT

$$\sin(\alpha-\beta) = \sin(\alpha+(-\beta)) = \sin(\alpha)\cos(-\beta) + \cos(\alpha)\sin(-\beta)$$
$$= \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta). \quad \square$$

THESE LEAD TO

THEOREM (SUM/DIFFERENCE RULE FOR TANGENTS)

$$\tan(\alpha\pm\beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$$

PROOF THIS IS LEFT AS AN EXERCISE FOR THE READER. IT FOLLOWS FROM $\tan\theta = \frac{\sin\theta}{\cos\theta}$ AND THE SUM/DIFFERENCE FORMULAS FOR SINE AND COSINE.

Ex $\cos(15^\circ) = \cos(60^\circ - 45^\circ)$

$$= \cos(60^\circ)\cos(45^\circ) + \sin(60^\circ)\sin(45^\circ)$$
$$= \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

Ex $\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{5\pi}{12}\right) + \cos\left(\frac{\pi}{12}\right)\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{12} + \frac{5\pi}{12}\right) = \sin\left(\frac{6\pi}{12}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

Section 5.3

Theorem (Double-Angle Formulas)

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

Proof Using the angle sum/difference formulas,

$$\sin(2\theta) = \sin(\theta+\theta) = \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta) = 2\cos(\theta)\sin(\theta).$$

The other two formulas are proven similarly. $\square$

Ex If $\sin\theta = \frac{3}{5}$ and θ in quadrant I, then $\cos\theta = \frac{4}{5}$ and so

$$\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\tfrac{3}{5}\right)\left(\tfrac{4}{5}\right) = \frac{24}{25}.$$

Notice that, applying a Pythagorean identity, we get

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1 \\ &= (1-\sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta.\end{aligned} \qquad (\star)$$

This leads us to:

Theorem (Power-Reducing Formulas)

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$
$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$
$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

Proof Rearrange the equations in $(\star)$ above. $\square$

Ex $\cos^2\left(\frac{\pi}{8}\right) = \frac{1+\cos\left(2\frac{\pi}{8}\right)}{2} = \frac{1+\cos(\pi/4)}{2} = \frac{1+\frac{\sqrt{2}}{2}}{2} = \frac{2+\sqrt{2}}{4}$

THEOREM (HALF-ANGLE FORMULAS)

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos(\alpha)}{2}}$$
$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos(\alpha)}{2}}$$
$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos(\alpha)}{1+\cos(\alpha)}}$$

PROOF LET $\theta = \frac{\alpha}{2}$ IN THE POWER REDUCING FORMULAS. $\square$

Ex $\sin(22.5^\circ) = \sin\left(\frac{45^\circ}{2}\right) = \pm\sqrt{\frac{1-\cos(45^\circ)}{2}} = \pm\sqrt{\frac{1-\sqrt{2}/2}{2}}$

$= \pm\sqrt{\frac{2-\sqrt{2}}{4}}$ SINCE $\sin\theta > 0$,

$= \frac{\sqrt{2-\sqrt{2}}}{2}$

SECTION 5.4

THEOREM (PRODUCT-TO-SUM FORMULAS)

$$\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$
$$\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha-\beta) + \cos(\alpha+\beta)\right)$$
$$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$$
$$\cos\alpha\sin\beta = \frac{1}{2}\left(\sin(\alpha+\beta) - \sin(\alpha-\beta)\right)$$

PROOF $\cos(\alpha-\beta) - \cos(\alpha+\beta) = (\cos\alpha\cos\beta + \sin\alpha\sin\beta) - (\cos\alpha\cos\beta - \sin\alpha\sin\beta)$

$= 2\sin\alpha\sin\beta$

THE REST ARE PROVEN SIMILARLY. $\square$

THEOREM (SUM-TO-PRODUCT FORMULAS)

$$\sin\alpha \pm \sin\beta = 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)$$
$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$
$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

PROOF LET $\theta = \frac{\alpha+\beta}{2}$, $\psi = \frac{\alpha-\beta}{2}$. THEN

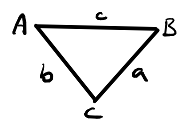
$$
\begin{aligned}
2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) &= 2\sin\theta\sin\psi \\
&= 2\left(\frac{1}{2}\left[\sin(\theta+\psi) + \sin(\theta-\psi)\right]\right) \quad \text{(product-to-sum formula)} \\
&= \sin(\theta+\psi) + \sin(\theta-\psi), \\
&= \sin\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right) + \sin\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\right) \\
&= \sin\alpha + \sin\beta.
\end{aligned}
$$

THE OTHERS FOLLOW SIMILARLY.

Section 6.1

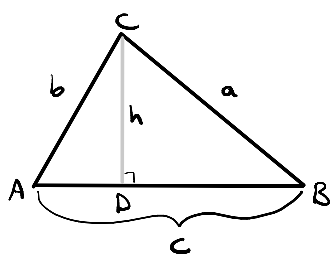
We know how to apply trig to solve right triangles, but the world is not all right triangles. Luckily, all hope is not lost.

Theorem (Law of Sines) Consider the general triangle:



Then $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

Proof



Notice that we can drop an altitude and create two right triangles. Then

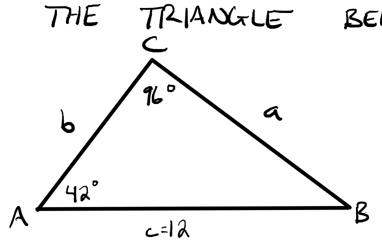
$\sin B = \dfrac{h}{a} \Rightarrow h = a\sin B$, and

$\sin A = \dfrac{h}{b} \Rightarrow h = b\sin A$. So

$$a\sin B = b\sin A$$
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

By dropping an altitude from B to $\overline{AC}$, we achieve $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$, hence the desired result. □

Ex Solve the triangle below:

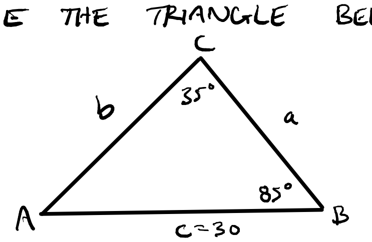


$B = 180° - 96° - 42° = 42°$

$a = \dfrac{c\sin A}{\sin C} = \dfrac{12\sin(42°)}{\sin(96°)} = 8.07$

$b = \dfrac{c\sin B}{\sin C} = \dfrac{12\sin(42°)}{\sin(96°)} = 8.07$

Ex Solve the triangle below



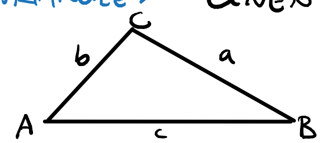
$A = 180° - 85° - 35° = 60°$

$b = \dfrac{c\sin B}{\sin C} = \dfrac{30\sin(85°)}{\sin(35°)} = 52.10$

$a = \dfrac{c\sin A}{\sin C} = \dfrac{30\sin(60°)}{\sin(35°)} = 45.30$

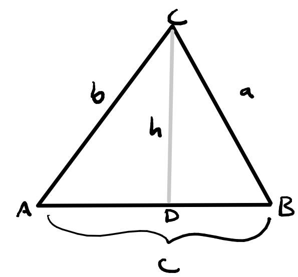
Trig also allows us to figure out the area of a general triangle.

Theorem (Area of a Triangle) Given the triangle below,



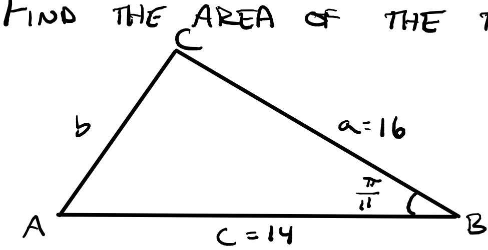
The area of this triangle is: $A = \frac{1}{2} bc \sin A = \frac{1}{2} ac \sin B = \frac{1}{2} ab \sin C$.

Proof Recall that the area of a triangle is given by $\frac{1}{2} \cdot \text{base length} \cdot \text{height}$.



Since $\sin A = \frac{h}{b}$, $h = b \sin A$, so
$\text{Area} = \frac{1}{2} bc \sin A$. Repeat this for the other sides. $\square$

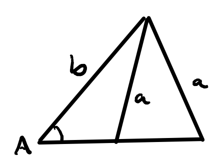
Ex Find the area of the triangle below:

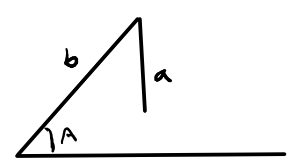


$\text{Area} = 16(14)\sin\left(\frac{\pi}{11}\right) = 63.11$

Ambiguous Triangles (SSA)

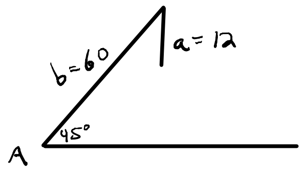
Two adjacent sides and an angle, like in the figures to the right, does not necessarily define a unique triangle.





Ex SOLVE TRIANGLE ABC, WHERE $A = 45°$, $a = 12$, $b = 60$.

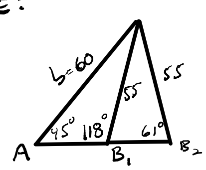
$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \sin B = \frac{b \sin A}{a} = \frac{60 \sin(45°)}{12} = 3.53,$$ WHICH IS NOT IN THE RANGE OF SINE. INDEED, THE PICTURE LOOKS SOMETHING LIKE THIS



Ex SOLVE TRIANGLE ABC IF $A = 45°$, $a = 55$, $b = 60$.

$$\sin B = \frac{b \sin A}{a} = \frac{60 \sin(45°)}{55} \approx 0.77,$$

SO $B \approx 61.77°$ OR $118.23°$, SO EITHER PICTURE BELOW IS POSSIBLE:

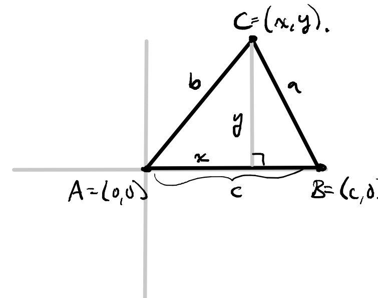


IN FACT, UNLESS $B = 90°$ (OR IF $a = b$, ASSUMING WE ONLY CONSIDER NONDEGENERATE CASES), THE TRIANGLE EITHER DOES NOT EXIST OR IS NOT UNIQUE.

Section 6.2

Theorem (Law of Cosines) If ABC is a triangle with opposite sides of lengths a, b, c (resp.), then

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$



Proof Placing our triangle in the Cartesian plane (as pictured to the right), we have

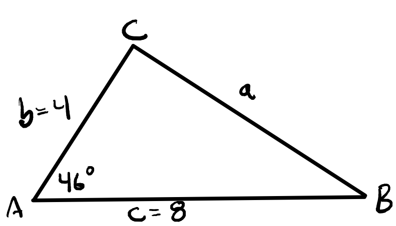
$$\cos A = \frac{x}{b} \Rightarrow x = b\cos A$$
$$\sin A = \frac{y}{b} \Rightarrow y = b\sin A.$$

By the distance formula, we have that

$$a = \sqrt{(x-c)^2 + (y-0)^2}$$
$$a^2 = (x-c)^2 + y^2$$
$$= x^2 - 2xc + c^2 + y^2$$
$$= b^2\cos^2 A - 2bc\cos A + c^2 + b^2\sin^2 A$$
$$= b^2(\cos^2 A + \sin^2 A) + c^2 - 2bc\cos A$$
$$= b^2 + c^2 - 2bc\cos A.$$

Playing this same game for the other two sides provides the result.

Ex

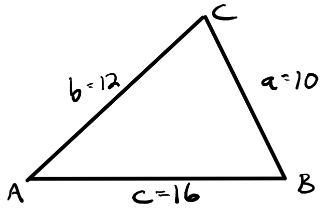


$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$a = \sqrt{16 + 64 - 64\cos(46^\circ)} = 5.96$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$\Rightarrow \cos B = \frac{a^2 + c^2 - b^2}{2ac} \approx 0.8759$$
$$\Rightarrow B \approx 28.85^\circ$$

$$C = 180^\circ - A - B = 180^\circ - 46^\circ - 28.85^\circ = 105.15^\circ$$

Ex



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\Rightarrow \cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{144+256-100}{2(12)(16)} = \frac{25}{32}$$

$$\Rightarrow A \approx 38.62°$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$\Rightarrow \cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{100+256-144}{2(10)(16)} = \frac{53}{80}$$

$$\Rightarrow B \approx 48.51°$$

$$C = 180° - A - B = 180° - 38.62° - 48.51° = 92.87°$$

Theorem (Heron's Formula) Let a, b, c be the side lengths of a triangle. Then

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{1}{2}(a+b+c)$.

Proof By the half-angle formula,

$$\cos\left(\frac{C}{2}\right) = \sqrt{\frac{1+\cos C}{2}} = \sqrt{\frac{1+\frac{a^2+b^2-c^2}{2ab}}{2}}$$

Law of cosines, solve for cos C

$$= \sqrt{\frac{2ab+a^2+b^2-c^2}{4ab}}$$

$$= \sqrt{\frac{(a+b)^2-c^2}{4ab}}$$

$$= \sqrt{\frac{((a+b)+c)((a+b)-c)}{4ab}}$$

Now let $s = \frac{1}{2}(a+b+c)$, so that $a+b+c = 2s$. Similarly,

$$a+b-c = a+b+c-2c = 2s-2c = 2(s-c),$$ so

$$= \sqrt{\frac{(2s)2(s-c)}{4ab}} = \sqrt{\frac{s(s-c)}{ab}}$$

BY A SIMILAR PROCESS, WE OBTAIN

$$\sin\left(\frac{C}{2}\right) = \sqrt{\frac{1-\cos C}{2}} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

RECALL NOW THAT $\text{Area} = \frac{1}{2}ab\sin C$. So,

$$\text{Area} = \frac{1}{2}ab\sin C = \frac{1}{2}ab \cdot 2\sin\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)$$

$$= ab\left(\sqrt{\frac{(s-a)(s-b)}{ab}}\right)\left(\sqrt{\frac{s(s-c)}{ab}}\right)$$

$$= ab\sqrt{\frac{s(s-a)(s-b)(s-c)}{(ab)^2}}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

Ex FOR THE PREVIOUS TRIANGLE, $a=10$, $b=12$, $c=16$. So $s = \frac{1}{2}(10+12+16) = 19$, HENCE

$$\text{Area} = \sqrt{19(19-10)(19-12)(19-16)} \approx 59.92 \text{ un}^2.$$