Mapping Class Groups and Classifying Surface Automorphisms

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The fundamental group of a topological space is a familiar and important algebraic invariant. Somewhat less familiar, but equally as important, is the mapping class group of a topological space. In this talk, we'll explore this algebraic invariant for surfaces (so that the lecturer can actually draw meaningful pictures) and look at the work of Nielsen and Thurston in classifying surface automorphisms.

The main references this talk are *A Primer on Mapping Class Groups* by B. Farb and D. Margalit, and *Automorphisms of Surfaces after Nielsen and Thurston* by S. Bleiler and A. Casson.

Isotopy and Mapping Class Groups

Everything I'm about to say about mapping class groups is more generally defined, but for the aim of this talk, will be restricted to the category of compact oriented surfaces (so we allow for nonempty boundary).

Definition. An *isotopy* of a surface S is a homotopy $H : S \times I \to S$ such that, for each $t \in I$, $H : S \times \{t\} \to S$ is a homeomorphism. Two self-homeomorphisms (hereforto called *automorphisms*) $f, g : S \to S$ are *isotopic* if there exists an isotopy between them.

Definition. The *mapping class group* of S is the group

$$\mathrm{MCG}(S) = \mathrm{Homeo}^+(S, \partial S)/\text{isotopy},$$

where $\mathrm{Homeo}^+(S, \partial S)$ is the group of orientation-preserving automorphisms of S that fix ∂S. This is a topological group with the compact open topology.

In the literature, we may also see the definition that

$$\mathrm{MCG}(S) = \mathrm{Homeo}^+(S, \partial S)/\,\mathrm{Homeo}_0^+(S, \partial S),$$

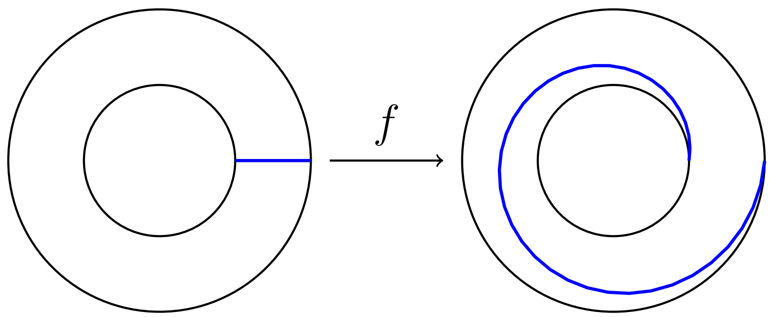
where $\mathrm{Homeo}_0^+(S, \partial S)$ is the path-component of the identity in $\mathrm{Homeo}^+(S, \partial S)$ (this is reasonable since path-components and isotopies coincide in the compact-open topology).

Example.

- $\mathrm{MCG}(S^2) = \mathbb{Z}/(2)$
- $\mathrm{MCG}(T^2) = \mathrm{SL}(2, \mathbb{Z})$
- $\mathrm{MCG}(\text{pair of pants}) = \mathbb{Z}^3$

Dehn Twists

Definition. Given an annulus $S^1 \times I$ parametrized via $(e^{i\theta}, t)$, a *Dehn twist* is a homeomorphism $f : S^1 \times I \to S^1 \times I$ given by $f(e^{i\theta}, t) = (ei(\theta + 2\pi t), t)$. Visually,

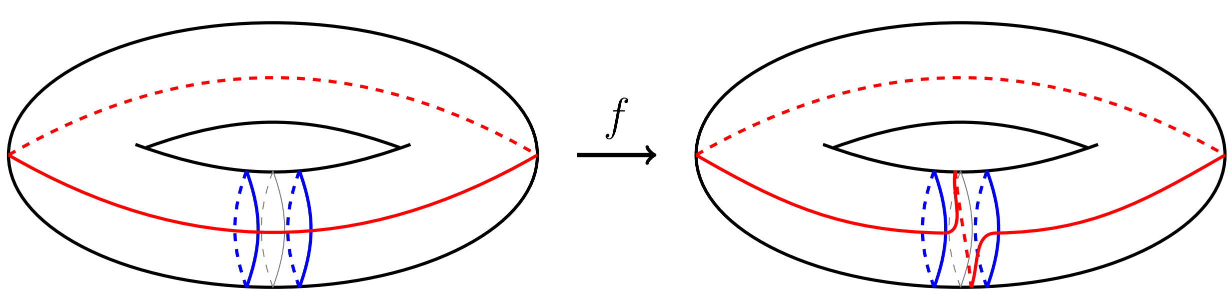


If c is a simple closed curve in our surface, a *Dehn twist about* c is a Dehn twist of an annular neighborhood of c.

Remark. We can extend Dehn twists to surface automorphisms in the following way: Let S be a surface, c a simple closed curve in S and N a (closed) annular neighborhood of c, and f a Dehn twist about c. Then the map $\varphi : S \to \Sigma$

$$\varphi(x) = \begin{cases} f(x) & \text{if } x \in N \\ x & \text{if } x \notin N \end{cases}$$

is a surface automorphism. Visually,



Remark (Fact). Dehn twists are nontrivial elements of MCG(S) and moreover have infinite order.

Example. Dehn twists of the longitudinal and latitudinal curves in the torus T^2 correspond to elements of $\mathrm{SL}(2,\mathbb{Z})$, namely

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

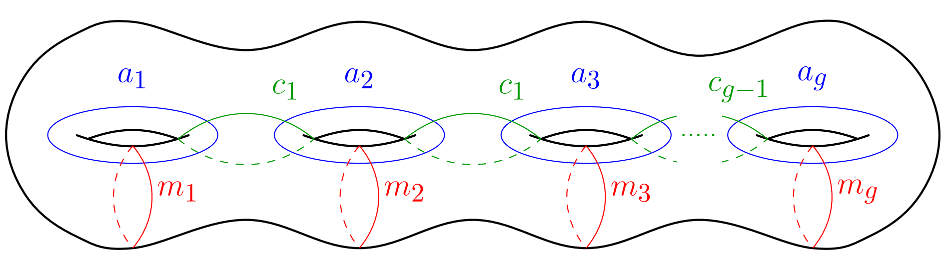
Remark. The keen eye will notice that the matrices above are generators for $\mathrm{SL}(2,\mathbb{Z}) = \mathrm{MCG}(T^2)$. This leads us to the following result.

Theorem 1 (Dehn-Lickorish)**.** *The mapping class group of a closed oriented surface of genus $g \geq 2$ is generated by finitely-many Dehn twists.*

In 1938, Dehn proved that $\mathrm{MCG}(S_g)$ could be generated by $2g(g-1)$ Dehn Twists.

In 1964, Lickorish, apparently unaware of Dehn's result, proved that, for $g \geq 2$, $\mathrm{MCG}(S_g)$ could be generated by $3g-1$ Dehn twists along nonseparating curves $a_1, \ldots, a_g, c_1, \ldots, c_{g-1}, m_1, \ldots, m_g$ as shown below. As such, these twists are called the *Lickorish generating set* (or *Lickorish twists*).

In 1979, Humphries improved Lickorish's result to show that, for $g \geq 2$, $\mathrm{MCG}(S_g)$ could be generated by $2g+1$ Dehn twists (the minimal number of twists) along nonseparating curves $a_1, \ldots, a_g, c_1 \ldots, c_{g-1}, m_1, m_2$ as shown below. As such, these twists are called the *Humphries generating set.*



Elements of $\mathrm{MCG}(S_g)$

For the remainder of this talk, S_g will be a closed orientable surface of genus $g \geq 2$.

Consider the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. As we said before $\mathrm{MCG}(T^2) = \mathrm{SL}(2, \mathbb{Z})$, so transformations $A \in \mathrm{SL}(2, \mathbb{Z})$ induce orientation-preserving automorphisms $h_A : T^2 \to T^2$. One of the following is true about A:

a. A has complex eigenvalues. In this case, A has finite order, so we call h_A *periodic.*

b. A has eigenvalues ± 1 of multiplicity 2. In this case, A has an eigenvector that descends to an essential simple closed curve (S.C.C.) in the quotient, so we call h_A *reducible* (it is a power of a Dehn twist).

c. A has distinct real eigenvalues $\lambda > 1$ and $\frac{1}{\lambda} < 1$. In this case, h_A is a linear homeomorphism stretching in one direction by a factor of λ and shrinking by the same factor in the complementary direction. We call h_A *Anosov.*

Question: Can we classify all automorphisms (up to isotopy) for any compact oriented surfaces of genus $g > 1$?

Extending Definitions

Definition. $h : S_g \to S_g$ is *periodic* if h^n is homotopic to the identity for some $n > 0$.

Remark. It turns out that we don't even need isotopy. A result of Fenchel and Nielsen says that if $h^n : S_g \to S_g$ is homotopic to the identity, then it is isotopic to a map $g : S_g \to S_g$ such that $g^n = \mathrm{Id}$.

Definition. $h : S_g \to S_g$ is *reducible* if it is homotopic to an automorphism that leaves invariant an essential closed 1-submanifold of S_g.

Remark. Here, an "essential" submanifold is one that does not bound a disk in the surface.

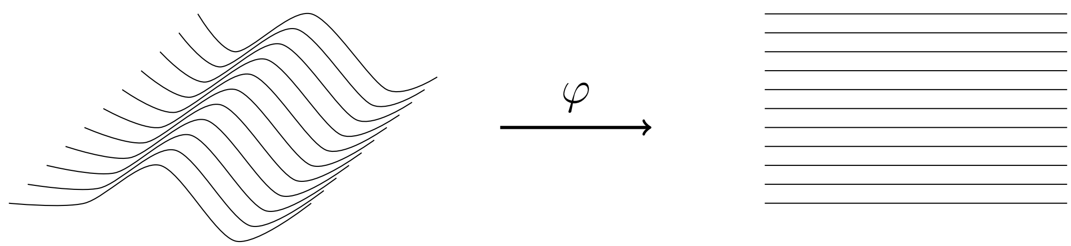
Question: These are clear, but how do we extend the notion of an Anosov map?

Thurston's Idea

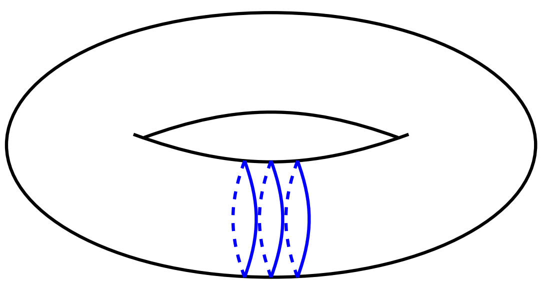
Definition. A curve is *simple* if it has no transverse self-intersections.

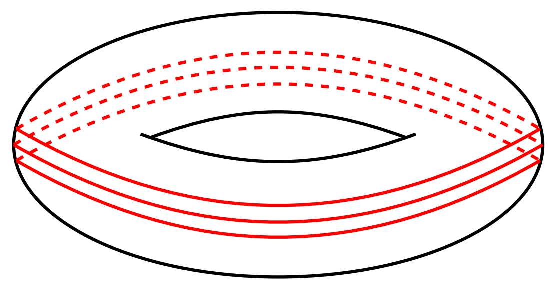
Definition. A *foliation* of S_g is a decomposition of S_g into a disjoint union of simple closed curves with no transverse self-intersections. These geodesics are called *leaves*.

Remark. Locally, using our manifold charts, our manifold looks like



Notice that the torus has two obvious foliations corresponding the lattitudinal and longitudinal circles

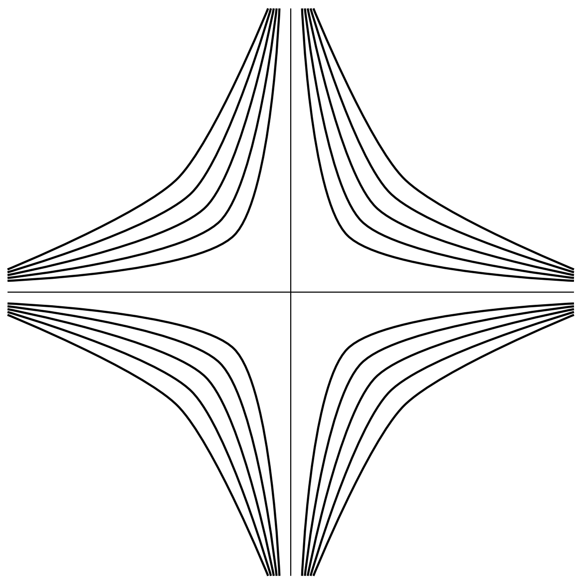




Each foliation corresponds to one of the scaling directions in the Anosov mapping, so maybe there is a way to re-interpret the definition in front of these foliations.

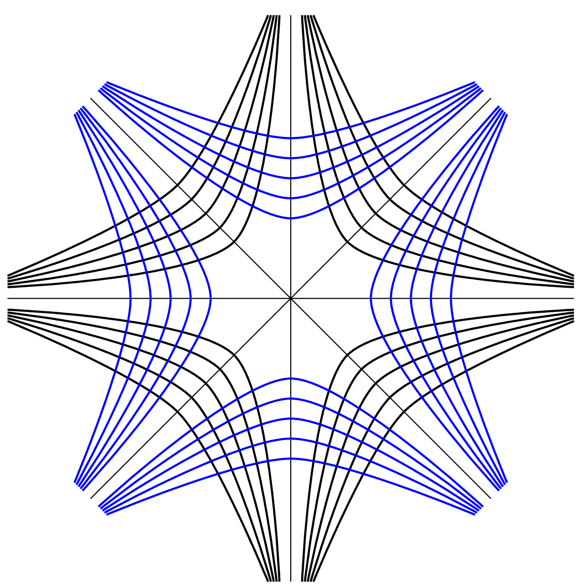
As a result of the Poincaré-Hopf theorem, genus $g \geq 2$ surfaces do not have such nice foliations, but come very close:

Definition. A *singular foliation* $\mathcal{F}$ of S_g is a foliation of $S_g - \mathrm{Sing}(S_g)$, where $\mathrm{Sing}(S_g)$ is a finite set of points $x \in S_g$ whose neighborhood charts have image the standard k-pronged singularity



These points comprise the *singular set* of $\mathcal{F}$, denoted $\mathrm{Sing}(\mathcal{F})$

Definition. Two singular foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are called *transverse* if $\mathrm{Sing}(\mathcal{F}_1) = \mathrm{Sing}(\mathcal{F}_2)$ and at all other points, the leaves are transverse.



Definition. A *transverse measure* μ *to a foliation* $\mathcal{F}$ *of* S_g is a Borel measure assigning a nonnegative real number to every path transverse to the leaves in our foliation, satisfying:

1. If c is the concatenation of paths c_1 and c_2, then $\mu(c) = \mu(c_1) + \mu(c_2)$
2. If c_1 and c_2 are homotopic paths, then $\mu(c_1) = \mu(c_2)$

So the leaves of the foliation have measure zero.

Definition. A *measured folation of* S_g is a pair $(\mathcal{F}, \mu)$ consisting of a singular foliation $\mathcal{F}$ and a transverse measure μ to $\mathcal{F}$.

Definition. An automorphism h is called *pseudo-Anosov* if there are transverse measured foliations $(\mathcal{F}_1, \mu_1)$ and $(\mathcal{F}_2, \mu_2)$ such that

$$(h(\mathcal{F}_1), h_*\mu_1) = (\mathcal{F}_1, \lambda\mu_1)$$
$$(h(\mathcal{F}_2), h_*\mu_2) = (\mathcal{F}_2, \frac{1}{\lambda}\mu_2)$$

for some $\lambda > 1$.

Moreover, this has the same effect as an Anosov map except on a set of measure zero on our surface. And this brings us to the main result:

Theorem 2 (Nielsen-Thurston)**.** *Any non-periodic irreducible automorphism of a closed oriented surface is isotopic to a pseudo-Anosov automorphism.*