Hybrid subgroups of complex hyperbolic lattices

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Arizona State University

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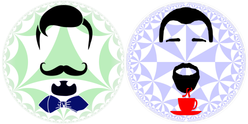

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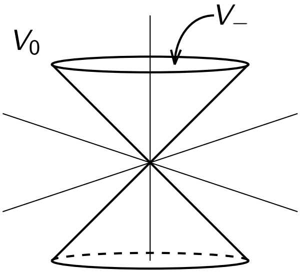

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$$H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

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(22 commensurability classes and 2 commensurability classes, resp.)

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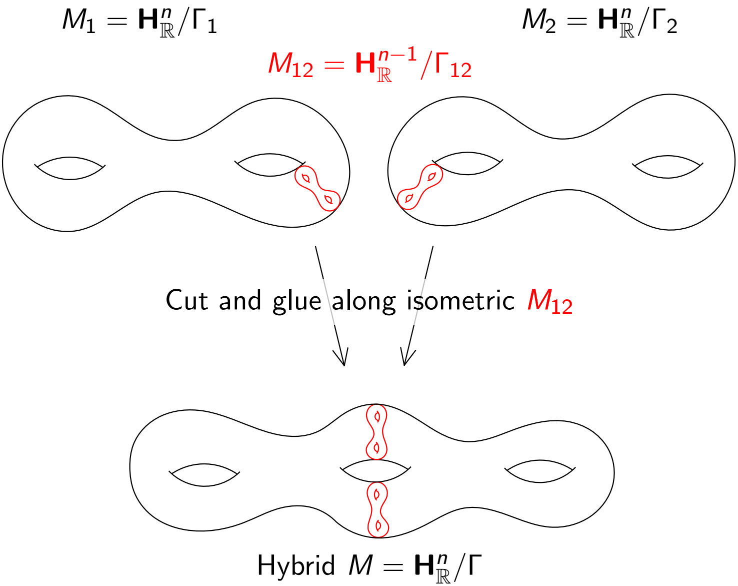

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Theorem (Gromov–Piatetski-Shapiro, '87)

If Γ_1 and Γ_2 are not commensurable, hybrid Γ is non-arithmetic.

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A *hybrid* of Γ_1 and Γ_2 is $H(\Gamma_1, \Gamma_2) := \langle \Lambda_1, \Lambda_2 \rangle < \mathrm{Isom}(\mathbf{H}^{n+1}_{\mathbb{C}})$.

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Let $H(d) = \langle \Lambda_1, \Lambda_2 \rangle$ denote the hybrid of Γ_1, Γ_2.

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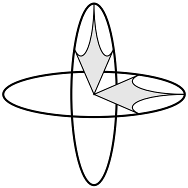

Embedding of fundamental domains for $\mathrm{SU}(1,1;\mathcal{O}_3)$ into $\mathbf{H}^2_{\mathbb{C}}$

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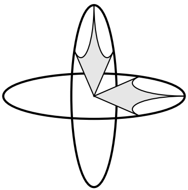

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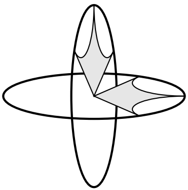

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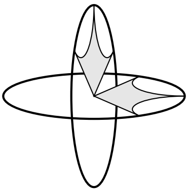

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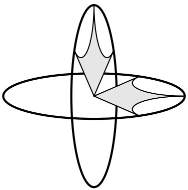

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3. *$H(7)$ is a lattice in Picard modular group* $\mathrm{PU}(2,1;\mathcal{O}_7)$*.*

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$\tilde{\Gamma}(p,t) < \mathrm{PU}(2,1)$ generated by three complex reflections, R_i, each of order p, and J, an order 3 element conjugating R_i to R_{i+1}.

Reflections satisfy the braid relation $R_i R_j R_i = R_j R_i R_j$.

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The following are non-arithmetic lattices: $\tilde{\Gamma}(3, 5/42)$, $\tilde{\Gamma}(3, 1/12)$, $\tilde{\Gamma}(3, 1/30)$, $\tilde{\Gamma}(4, 3/20)$, $\tilde{\Gamma}(4, 1/12)$, $\tilde{\Gamma}(5, 1/5)$, $\tilde{\Gamma}(5, 11/30)$.

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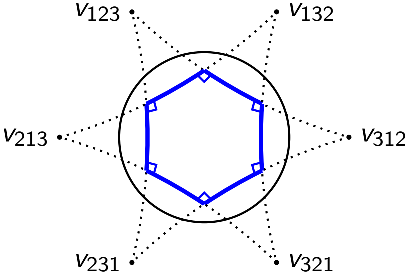
This introduces positive vectors v_{ijk} satisfying $v_{ijk} \perp v_{jik}$ and $v_{ijk} \perp v_{ikj}$



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Theorem (W,'18)

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$\tilde{\Gamma}(4, 1/12)$ *and* $\tilde{\Gamma}(5, 1/5)$ *are non-arithmetic and arise as hybrids of non-commensurable arithmetic Fuchsian triangle groups.*

Thank you.

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