Hairy Ball Theorem

Joseph Wells
Arizona State University

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Introduction to the Theorem

"You can't comb a hairy ball flat without creating a cowlick."

More mathematically,

Theorem

S^2 does not have a field of nonzero tangent vectors.

To prove this theorem, we'll need to build up some of the necessary (simplicial) homology first.

Basics of Homology

The idea of homology is that it counts the number of holes in our structure in each dimension.

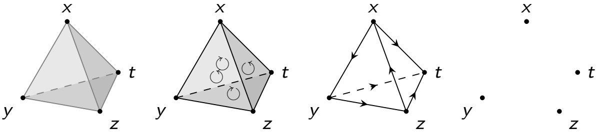
- The n^{th} chain group C_n is the free abelian group whose generators are the distinct n-dimensional objects.
- Given a set of vertices $\{v_0, \ldots, v_n\}$ and the n-dimensional structure $[v_0 \cdots v_n]$, then n^{th} bounday homomorphism $\partial_n : C_n \to C_{n-1}$ acts as follows:

$$\partial_n([v_0 \cdots v_n]) = \sum_{i=0}^{n} (-1)^i [v_0 \cdots \hat{v}_i \cdots v_n]$$

  where the hat vertex is omitted.
- n-Cycles are denoted $Z_n = \ker \partial_n$.
- n-Boundaries are denote $B_n = \operatorname{im} \partial_{n+1}$.
- The n^{th} homology group is the quotient $H_n = Z_n / B_n$.
- The reduced homology group $\tilde{H}_n = 0$ when $n = 0$ and $\tilde{H}_n = H_n$ otherwise.

Computing the Homology of S^2

Since S^2 is homeomorphic to the hollow tetrahedron, we can view our chain complex as follows:



$$C_3 \xrightarrow[\partial_3]{} C_2 \xrightarrow[\partial_2]{} C_1 \xrightarrow[\partial_1]{} C_0 \xrightarrow[\partial_0]{} 0$$

$$\begin{aligned} C_3 &= 0 & & \\ C_2 &= \langle [xyz], [xzt], [ytz], [xty] \rangle & C_2 &\cong \mathbb{Z}^4 \\ C_1 &= \langle [xy], [xz], [xt], [yz], [yt], [xt] \rangle & C_1 &\cong \mathbb{Z}^6 \\ C_0 &= \langle [x], [y], [z], [t] \rangle & C_0 &\cong \mathbb{Z}^4 \end{aligned}$$

Today we're only interested in calculating $H_2(S^2)$.

Since $Z_2 = \ker \partial_2 \lhd C_2$, elements are integral combinations of the four 2-simplices. Thus, for $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$,

$$\begin{aligned}
0 &= \partial_2 \left(\alpha[xyz] + \beta[xzt] + \gamma[ytz] + \delta[xty]\right) \\
&= \alpha\left([yz] - [xz] + [xy]\right) + \beta\left([zt] - [xt] + [xz]\right) \\
&\quad + \gamma\left([tz] - [yz] + [yt]\right) + \delta\left([ty] - [xy] + [xt]\right) \\
&= (\alpha - \delta)[xy] + (-\alpha + \beta)[xz] + (-\beta + \delta)[xt] \\
&\quad + (\alpha - \gamma)[yz] + (\gamma - \delta)[yt] + (\beta - \gamma)[zt]
\end{aligned}$$

We can write this as the following matrix

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

By row reduction, we get

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So our null space has dimension 1, thus $Z_2 = \ker \partial_2 \cong \mathbb{Z}$. Since $B_2 = \operatorname{im} \partial_3 \cong 0$, then $H_2(S^2) \cong \mathbb{Z}/0 \cong \mathbb{Z}$.

This makes sense geometrically too. The faces are a cycle for exactly 1 hole, so we can extrapolate this to see, somewhat intuitively, that $H_n(S^n) \cong \mathbb{Z}$ always.

Degree Map

A continuous map $f : S^n \to S^n$, induces a homomorphism on the (reduced) homology $f_* : \tilde{H}_n(S^n) \to \tilde{H}_n(S^n)$.

Since f_* maps an infinite cyclic group to an infinite cyclic group, so it must be of the form $f_*(x) = nx$, where $n \in \mathbb{Z}$.

We call this integer n the *degree* of f.

Properties of Degree Map

- $\deg(\mathbf{1}) = 1$ since $\mathbf{1}_* = \mathbf{1}$.
- $\deg(fg) = \deg(f)\deg(g)$ since $(fg)_* = f_* g_*$
- If $f \simeq g$, then $\deg(f) = \deg(g)$.
- $\deg(f) = -1$ if f is a reflection - if we fix a subsphere S^{n-1} and interchange complementary hemispheres.
- $\deg(-\mathbf{1}) = (-1)^{n+1}$ where the $-\mathbf{1} : S^n \to S^n$ via $x \mapsto -x$ is the antipodal map, as it is the composition of $(n+1)$-many reflections.

Theorem (Hat2.28)

S^n has a continuous field of nonzero tangent vectors iff n is odd.

Proof.

Suppose we have the tangent vector field on S^n given by $x \mapsto v(x)$, where $x \in S^n$ and $v(x) \in T_x S^n$. In fact, if $v(x) \neq 0$, we may as well just assume that $v(x) \in U_x S^n$. Consider the great circle in the plane spanned by x and $v(x)$: $x \sin(t) + v(x) \cos(t)$. Then $F_t : S^n \to S^n$ given by $F_t(x) = x \sin(t) + v(x) \cos(t)$, $t \in [0, \pi]$ is a homotopy from the identity map $\mathbf{1}$ to $-\mathbf{1}$, thus $\deg(\mathbf{1}) = \deg(-\mathbf{1})$, so $(-1)^{n+1} = 1$, and therefore n is odd.

Conversely, suppose that $n = 2k - 1$. Then for each $x = (x_1, \dots, x_{2k})$, define $v(x) = (-x_2, x_1, \dots, -x_{2k}, x_{2k-1})$. Then $v(x)$ is orthogonal to x and $v(x) \neq 0$. □

Corollary (Hairy Corollary)

S^2 does not have a field of nonzero tangent vectors.

Proof.

2 is not odd. □

Thank you.