Non-arithmetic hybrid lattices in $\mathrm{PU}(2,1)$

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October 26, 2020

Abstract

We explore hybrid subgroups of certain non-arithmetic lattices in $\mathrm{PU}(2,1)$. In particular, we show that all of Mostow's lattices are virtually hybrids; moreover, we show that some of these non-arithmetic lattices are virtually hybrids of two non-commensurable arithmetic lattices in $\mathrm{PU}(1,1)$.

1 Introduction

One key notion in the study of lattices in a semisimple Lie group G is that of *arithmeticity* (which we will not define here; see [9] for a standard reference). When G arises as the isometry group of a symmetric space X of non-compact type, the combined work of Margulis [8], Gromov–Schoen [6], and Corlette [1] imply that non-arithmetic lattices only exist when $X = \mathbf{H}^n_{\mathbb{R}}$ or $\mathbf{H}^n_{\mathbb{C}}$ (real and complex hyperbolic space, respectively); equivalently, up to finite index, when $G = \mathrm{PO}(n,1)$ or $\mathrm{PU}(n,1)$. Due to their exceptional nature, it has been a major challenge to find and understand non-arithmetic lattices in these Lie groups.

Given two arithmetic lattices Γ_1, Γ_2 in $\mathrm{PO}(n,1)$ with common sublattice $\Gamma_{1,2} \leq \mathrm{PO}(n-1,1)$, Gromov and Piatestki-Shapiro showed in [5] that one can produce a new "hybrid" lattice Γ in $\mathrm{PO}(n,1)$ by way of a technique that they call "interbreeding" or "hybridization". In particular, when Γ_1 and Γ_2 are not commensurable, Γ is shown to be non-arithmetic. It has been asked whether an analogous technique can exist for lattices in $\mathrm{PU}(n,1)$.

In [12], Paupert explores a possible analog (which he attributes to unpublished work of Hunt) where one starts with two arithmetic lattices Γ_1, Γ_2 in $\mathrm{PU}(n,1)$ and embeddings $\iota_i : \mathrm{PU}(n,1) \hookrightarrow \mathrm{PU}(n+1,1)$ such that (1) $\iota_1(\Gamma_1)$ and $\iota_2(\Gamma_2)$ stabilize totally geodesic complex hypersurfaces in $\mathbf{H}^{n+1}_{\mathbb{C}}$, (2) these hypersurfaces are orthogonal to one another, and (3) $\iota_1(\Gamma_1) \cap \iota_2(\Gamma_2)$ is a lattice in $\mathrm{PU}(n-1,1)$. The hybrid subgroup is then $H(\Gamma_1,\Gamma_2) := \langle \iota_1(\Gamma_1), \iota_2(\Gamma_2) \rangle$.

Using the above construction, Paupert then produces an infinite family of hybrids that are non-discrete in [12]. Following this, in [13], Paupert and the author used the same hybridization technique to produce both arithmetic lattices and thin subgroups of the Picard modular groups. In Section 2.2 here, we introduce a more general hybrid construction and explore it in the context of the lattices $\Gamma(p,t) \subset \mathrm{PU}(2,1)$ originally produced by Mostow in [10] (see Section 3 for explanation of notation). We obtain the following main results:

Theorem. *1. All of Mostow's lattices* $\Gamma(p,t)$ *are virtually hybrids.*

2. The non-arithmetic lattices $\Gamma(3,1/12)$, $\Gamma(4,1/12)$, *and* $\Gamma(5,1/5)$ *are virtually hybrids of two non-commensurable arithmetic lattices in* $\mathrm{PU}(1,1)$.

The second part of this theorem highlights the similarity of these hybrids and those hybrids of Gromov–Piatetski-Shapiro, specifically in that the hybridization procedure can produce a non-arithmetic lattice from two noncommensurable arithmetic lattices. We were unable to obtain all of the non-arithmetic lattices in Mostow's list from hybridizing two non-commensurable arithmetic lattices, as it is difficult to find candidate hypersurfaces.

In sections 4 and 5, for some values of (p,t) we obtain some commensurable (but possibly non-isomorphic) lattices by hybridizing using different pairs of hypersurfaces. While the setup and proofs used here rely upon an initial choice of a pair of orthogonal hypersurfaces, it's not clear that the resulting lattice is particularly sensitive to such choices, nor is it clear that every hybrid appearing in this work can be obtained in two different ways. We also note that there are more non-arithmetic lattices in $\mathrm{PU}(2,1)$ than those appearing in Mostow's list (see the survey [11] and the newer lattices found in [3]); it would be interesting to know if any of these lattices are (virtually) hybrids.

The author would like to thank Julien Paupert and the anonymous referee for many insightful comments and suggestions. Thank you also to Irene Pasquinelli for pointing out that there was a small error in the list of hybrids in the main theorem.

2 Complex hyperbolic geometry and hybrids

We give a brief overview of relevant definitions in complex hyperbolic geometry; the reader can see [4] for a standard source.

Let H be a Hermitian matrix of signature $(n,1)$ and let $\mathbb{C}^{n,1}$ denote $\mathbb{C}^{n+1}$ endowed with the Hermitian form $\langle\cdot,\cdot\rangle$ coming from H. Let V_- denote the set of points $z\in\mathbb{C}^{n,1}$ for which $\langle z,z\rangle<0$, and let V_0 denote the set of points for which $\langle z,z\rangle=0$. Given the usual projectivization map $\mathbb{P}:\mathbb{C}^{n,1}-\{0\}\to\mathbb{CP}^n$, *complex hyperbolic n-space* is $\mathbf{H}^n_{\mathbb{C}}:=\mathbb{P}(V_-)$ with distance d coming from the Bergman metric

$$\cosh^2\frac{1}{2}d(\mathbb{P}(x),\mathbb{P}(y))=\frac{|\langle x,y\rangle|^2}{\langle x,x\rangle\langle y,y\rangle}$$

The ideal boundary $\partial_\infty\mathbf{H}^n_{\mathbb{C}}$ is then identified with $\mathbb{P}(V_0)$.

2.1 Complex hyperbolic isometries

Let $\mathrm{U}(n,1)$ denote the group of unitary matrices preserving H. The holomorphic isometry group of $\mathbf{H}^n_{\mathbb{C}}$ is $\mathrm{PU}(n,1)=\mathrm{U}(n,1)/\mathrm{U}(1)$, and the full isometry group is generated by $\mathrm{PU}(n,1)$ and the antiholomorphic involution $z\mapsto\overline{z}$. Any holomorphic isometry of $\mathbf{H}^n_{\mathbb{C}}$ is one of the following three types:

- *elliptic* if it has a fixed point in $\mathbf{H}^n_{\mathbb{C}}$.
- *parabolic* if it has exactly one fixed point in the boundary (and no fixed points in $\mathbf{H}^n_{\mathbb{C}}$).

- *loxodromic* if it has exactly two fixed points in the boundary (and no fixed points in $\mathbf{H}^n_{\mathbb{C}}$).

Given a vector $v \in \mathbb{C}^{n,1}$ with $\langle v, v\rangle > 0$ and a complex number ζ with unit modulus, the map

$$R_{v,\zeta}(x) : x \mapsto x + (\zeta - 1)\frac{\langle x, v\rangle}{\langle v, v\rangle} v$$

is an an isometry of $\mathbf{H}^n_{\mathbb{C}}$ called a *complex reflection*, and its fixed point set $v^\perp \subset \mathbf{H}^n_{\mathbb{C}}$ is a totally geodesic subspace called a $\mathbb{C}^{n-1}$*-plane* (or a *complex line* when $n = 2$). We refer to v as a *polar vector* for the subspace $\mathbb{P}(v^\perp) \cap \mathbf{H}^n_{\mathbb{C}}$; abusing notation slightly we will denote such a projective subspace simply by $v^\perp$.

2.2 Complex hyperbolic hybrid construction

The lack of totally geodesic real hypersurfaces in $\mathbf{H}^n_{\mathbb{C}}$ presents an issue in finding a suitable complex-hyperbolic analog of the Gromov–Piatetski-Shapiro hybrid groups. Hunt's initial idea (see [12] for the first published reference to this construction) required subgroups of $\mathrm{PU}(n, 1)$ that stabilize an orthogonal hypersurface (on which it acts as a lattice in $\mathrm{PU}(n-1, 1)$) and fix the orthogonal complement. This second requirement, while convenient algebraically, is possibly too restrictive geometrically, and so we present below a modified construction in which this restriction is relaxed.

Definition. Let $\Gamma_1, \Gamma_2 < \mathrm{PU}(n, 1)$ be lattices. We define a *hybrid of* Γ_1, Γ_2 to be any group $H(\Gamma_1, \Gamma_2)$ generated by discrete subgroups $\Lambda_1, \Lambda_2 < \mathrm{PU}(n+1, 1)$ stabilizing totally geodesic hypersurfaces Σ_1, Σ_2 (respectively) such that

1. Σ_1 and Σ_2 are orthogonal,
2. $\Gamma_i = \Lambda_i|_{\Sigma_i}$, and
3. $\Lambda_1 \cap \Lambda_2$ is a lattice in $\mathrm{PU}(n-1, 1)$.

Remark. The groups explored by Paupert [12] and Paupert–Wells [13] are still hybrids in this new sense as well.

3 Mostow's lattices

In [10], Mostow constructed the first known non-arithmetic lattices in $\mathrm{PU}(2,1)$ among a family of groups generated by complex reflections. These groups, denoted $\Gamma(p,t)$, are defined as follows: Let $p = 3, 4, 5$, t be a real number satisfying $|t| < 3\left(\frac{1}{2} - \frac{1}{p}\right)$, $\alpha = \frac{1}{2\sin(\pi/p)}$, $\varphi = e^{\pi i t/3}$, and $\eta = e^{\pi i/p}$. Define a Hermitian form $\langle x, y\rangle = x^T H \overline{y}$ where

$$H = \begin{pmatrix} 1 & -\alpha\varphi & -\alpha\overline{\varphi} \\ -\alpha\overline{\varphi} & 1 & -\alpha\varphi \\ -\alpha\varphi & -\alpha\overline{\varphi} & 1 \end{pmatrix}.$$

For any pair (p,t) as above, the group $\Gamma(p,t)$ is generated by the three complex reflections of order p,

$$R_1 = \begin{pmatrix} \eta^2 & -i\eta\overline{\varphi} & -i\eta\varphi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ -i\eta\varphi & \eta^2 & -i\eta\overline{\varphi} \\ 0 & 0 & 1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -i\eta\overline{\varphi} & -i\eta\varphi & \eta^2 \end{pmatrix},$$

and these reflections satisfy the braid relations $R_iR_jR_i = R_jR_iR_j$. The mirror for the reflection R_i is given by $e_i^{\perp}$ where e_i is the standard i^{th} basis vector. When $|t| < \frac{1}{2} - \frac{1}{p}$, Mostow refers to these groups as having *small phase shift*. Similarly, when $|t| = \frac{1}{2} - \frac{1}{p}$ we'll refer to $\Gamma(p,t)$ as having *critical phase shift* and $|t| > \frac{1}{2} - \frac{1}{p}$ as having *large phase shift*. Since the groups $\Gamma(p,t)$ and $\Gamma(p,-t)$ are isomorphic, we restrict our focus to the cases where $t \geq 0$.

Remark (Tables 1 and 2 in [10]). For $p = 3, 4, 5$, there are only finitely-many values of t for which $\Gamma(p,t)$ is discrete, and they are given in Table 1. If $\Gamma(p,t)$ is discrete, we'll refer to the pair (p,t) as *admissible*.

p	$t < 1/2 - 1/p$	$t = 1/2 - 1/p$	$t > 1/2 - 1/p$
3	0, 1/30, 1/18, 1/12, 5/42	1/6	7/30, 1/3
4	0, 1/12, 3/20	1/4	5/12
5	1/10, 1/5		11/30, 7/10

Table 1: Values of p and t for which $\Gamma(p,t)$ is discrete.

Theorem 1 (Theorem 17.3 in [10]). *For each admissible pair (p,t), the group $\Gamma(p,t)$ is a lattice in* $\mathrm{PU}(2,1)$*, and the following are non-arithmetic:* $\Gamma(3,5/42)$, $\Gamma(3,1/12)$, $\Gamma(3,1/30)$, $\Gamma(4,3/20)$, $\Gamma(4,1/12)$, $\Gamma(5,1/5)$.

Remark. In Mostow's original list, $(5,11/30)$ was included as a non-arithmetic lattice, but in fact it is arithmetic (see Parker's survey [11, p. 27]).

Following the notation in [2], we examine closely related groups $\tilde{\Gamma}(p,t) = \langle R_1, J\rangle$ where

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

J has order 3 and $R_{i+1} = JR_iJ^{-1}$ (where $i = 1,2,3$ and indices are taken modulo 3). It is sufficient to study these groups $\tilde{\Gamma}(p,t)$ due to the following result:

Proposition 2 (Lemma 16.1 in [10]). *For each admissible pair (p,t), the group $\Gamma(p,t)$ has index dividing 3 in $\tilde{\Gamma}(p,t)$. The two groups are equal precisely when $k = \frac{1}{2} - \frac{1}{p} - \frac{1}{t}$ and $\ell = \frac{1}{2} - \frac{1}{p} + \frac{1}{t}$ are both integers and 3 does not divide both k and ℓ.*

4 Hybrids in Mostow's lattices

In this section, we'll construct hybrids from suitably-chosen hypersurfaces in the fundamental domains described by Deraux, Falbel, and Paupert in [2]. The reader is not expected to be familiar with the contents of this work, and we'll begin by reviewing all relevant information.

In [2], the authors found new fundamental domains for Mostow's lattices in each of the small, critical, and large phase shift cases (which have very different combinatorial structures). They first show that when $\tilde{\Gamma}(p,t)$ has small phase shift, a fundamental domain for this group can be constructed by coning over two polytopes that intersect in a right-angled hexagon, which Deraux–Falbel–Parker refer to as the "core" hexagon. Each side of this hexagon is contained in a complex line that is polar to a positive vector

v_{ijk} (see Figure 1), and taking lifts to $\mathbb{C}^{2,1}$ these vectors are given explicitly below:

$$v_{123} = \begin{pmatrix} -i\eta\overline{\varphi} \\ 1 \\ i\overline{\eta}\varphi \end{pmatrix}, \qquad v_{231} = \begin{pmatrix} i\overline{\eta}\varphi \\ -i\eta\overline{\varphi} \\ 1 \end{pmatrix}, \qquad v_{312} = \begin{pmatrix} 1 \\ i\overline{\eta}\varphi \\ -i\eta\overline{\varphi} \end{pmatrix},$$
$$v_{321} = \begin{pmatrix} i\overline{\eta\varphi} \\ 1 \\ -i\eta\varphi \end{pmatrix}, \qquad v_{132} = \begin{pmatrix} -i\eta\varphi \\ i\overline{\eta\varphi} \\ 1 \end{pmatrix}, \qquad v_{213} = \begin{pmatrix} 1 \\ -i\eta\varphi \\ i\overline{\eta\varphi} \end{pmatrix}.$$

Geometrically, $v_{ijk}^{\perp}$ is the mirror for the complex reflection $J^{\pm 1}R_jR_k$ for $k = i\pm 1 \pmod 3$. When $\tilde{\Gamma}(p,t)$, has critical phase shift, the fundamental domain changes and the core hexagon degenerates into an ideal triangle (see Figure 2) and JR_jR_k is parabolic (hence $\tilde{\Gamma}(p,t)$ is non-cocompact). When $\tilde{\Gamma}(p,t)$ has large phase shift, the fundamental domain changes yet again – the ideal vertices sit inside $\mathbf{H}^2_{\mathbb{C}}$ (see Figure 3) and JR_jR_k is elliptic. Section 6 of [2] discusses the new combinatorial structure of the fundamental domain in both the critical and large phase shift cases; however, here we're only concerned with this core polygon. It's worth noting that Mostow was also aware of this particular hexagon and its geometry in each of the different phase shift cases (see Section 9 of [10]), but his original fundamental domains were constructed in a very different fashion.

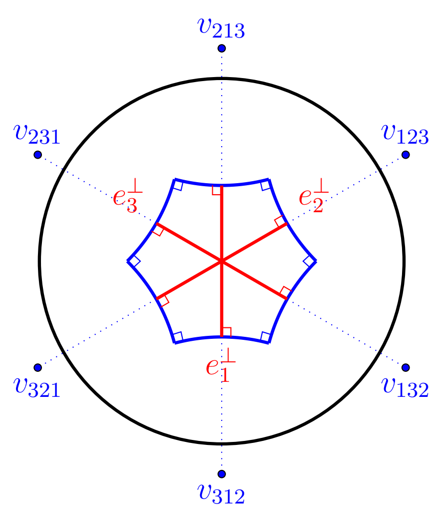

Figure 1: Core polygon when $0 \leq t < \frac{1}{2} - \frac{1}{p}$

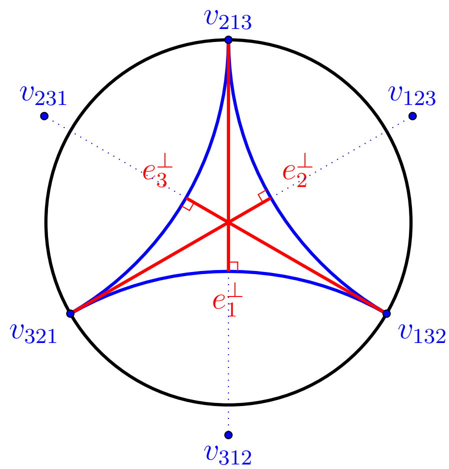

Figure 2: Core polygon when $t = \frac{1}{2} - \frac{1}{p}$

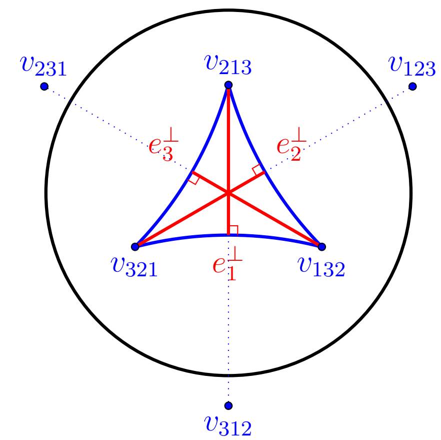

Figure 3: Core polygon when $t > \frac{1}{2} - \frac{1}{p}$

Recall that our hybrid construction requires two orthogonal hypersurfaces and discrete subgroups of $\mathrm{PU}(2,1)$ which stabilize them. The following readily-checked results give candidate subspaces:

Proposition 3 (Proposition 2.13(3) in [2])**.** $v_{ijk} \perp v_{jik}$ *and* $v_{ijk} \perp v_{ikj}$.

Proposition 4. *For the standard basis vectors* e_i*, we have* $e_i \perp v_{jik}$ *and* $e_i \perp v_{kij}$.

In the case of a critical phase shift, the vectors $v_{132}, v_{213}, v_{321}$ are null, and in the case of large phase shift, they are negative vectors. In both cases, the corresponding orthogonal projective subspaces do not intersect $\mathbf{H}^2_{\mathbb{C}}$. For this reason only in the case of small phase shift may we construct a hybrid from adjacent sides of the hexagon (which we explore separately in Section 5). However, in all phase shift cases, e_1 and v_{312} are positive vectors, hence they are polar to complex lines $e_1^{\perp}$ and $v_{312}^{\perp}$ in $\mathbf{H}^n_{\mathbb{C}}$ (similarly, for the pairs $e_2^{\perp}$, $v_{123}^{\perp}$ and $e_3^{\perp}$, $v_{231}^{\perp}$ as these lie in the same J-orbit as the pair $e_1^{\perp}$, $v_{312}^{\perp}$). As such, we use these two subspaces for our hybrid construction, which are written in homogeneous coordinates as

$$\begin{aligned} e_1^{\perp} &= \{[z, \varphi z/\alpha - \varphi^2, 1]^T \ : \ z \in \mathbb{C}\} \quad \text{and} \\ v_{312}^{\perp} &= \{[z, i\eta\overline{\varphi}, 1]^T \ : \ z \in \mathbb{C}\}. \end{aligned}$$

Let $\Lambda_{ijk} \leq \tilde{\Gamma}(p,t)$ be the subgroup stabilizing $v_{ijk}^{\perp}$ and let Λ_i be the subgroup stabilizing $e_i^{\perp}$. These groups are naturally identified with subgroups of $\mathrm{PU}(1,1)$, and so we let Γ_{ijk} and Γ_i be lifts of these groups (respectively) into $\mathrm{SU}(1,1)$. In this way, we see that $\Lambda_{ijk}|_{v_{ijk}^{\perp}} = \Gamma_{ijk}$ and $\Lambda_i|_{e_i^{\perp}} = \Gamma_i$, so we only need to check that Γ_{ijk} and Γ_i are indeed lattices.

Proposition 5. Γ_{312} *is a lattice in* $\mathrm{SU}(1,1)$*. It is cocompact for all non-critical phase shift values.*

Proof. v_{312} is a positive eigenvector for both R_1 and R_3J, hence they both stabilize $v_{312}^{\perp}$. The action of these elements on $v_{312}^{\perp}$ can be seen below:

$$R_1 : \begin{bmatrix} z \\ i\eta\overline{\varphi} \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} \eta^2 z + \overline{\varphi}^2 - i\eta\varphi \\ i\eta\overline{\varphi} \\ 1 \end{bmatrix}$$

$$R_3J : \begin{bmatrix} z \\ i\eta\overline{\varphi} \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} i\eta\overline{\varphi}/z \\ i\eta\overline{\varphi} \\ 1 \end{bmatrix}$$

Let A and B be the following elements in $\mathrm{SU}(1,1)$ corresponding to the actions of R_1 and R_3J on $v_{312}^{\perp}$, respectively,

$$A = \frac{1}{\eta}\begin{pmatrix} \eta^2 & \overline{\varphi}^2 - i\eta\varphi \\ 0 & 1 \end{pmatrix}, \qquad B = \frac{1}{\sqrt{-\overline{i\eta\overline{\varphi}}}}\begin{pmatrix} 0 & i\eta\overline{\varphi} \\ 1 & 0 \end{pmatrix}.$$

One then sees that

$$\begin{aligned} |\operatorname{Tr}(A)| &= \left|1 + e^{i2\pi/p}\right|, \\ |\operatorname{Tr}(B)| &= 0, \\ |\operatorname{Tr}(A^{-1}B)| &= \left|1 + e^{i\pi(t-1/2+1/p)}\right|. \end{aligned}$$

All of these values are less than or equal to 2 for all admissible pairs (p,t), so neither A nor B is loxodromic and thus they generate the orientation-preserving subgroup of a Fuchsian triangle group of finite covolume. It follows that Γ_{312} is a lattice in $\mathrm{PU}(1,1)$. By computing orders of these elements for admissible (p,t), one obtains Table 2 showing the corresponding triangle groups, and arithmeticity/non-arithmeticity (A/NA) of each can be checked by comparing with the main theorem of [14]. □

(p,t)	$\triangle(x,y,z)$	A/NA	(p,t)	$\triangle(x,y,z)$	A/NA
$(3,0)$	$\triangle(2,3,12)$	A	$(4,0)$	$\triangle(2,4,8)$	A
$(3,1/30)$	$\triangle(2,3,15)$	NA	$(4,1/12)$	$\triangle(2,4,12)$	A
$(3,1/18)$	$\triangle(2,3,18)$	A	$(4,3/20)$	$\triangle(2,4,20)$	NA
$(3,1/12)$	$\triangle(2,3,24)$	A	$(4,1/4)$	$\triangle(2,4,\infty)$	A
$(3,5/42)$	$\triangle(2,3,42)$	NA	$(4,5/12)$	$\triangle(2,4,12)$	A
$(3,1/6)$	$\triangle(2,3,\infty)$	A	$(5,1/10)$	$\triangle(2,5,10)$	A
$(3,7/30)$	$\triangle(2,3,30)$	A	$(5,1/5)$	$\triangle(2,5,20)$	A
$(3,1/3)$	$\triangle(2,3,12)$	A	$(5,11/30)$	$\triangle(2,5,30)$	A
			$(5,7/10)$	$\triangle(2,5,5)$	A

Table 2: Properties of Γ_{312}

Proposition 6. Γ_1 *is a lattice in* $\mathrm{SU}(1,1)$. *It is cocompact for all non-critical phase shift values.*

Proof. $J^{-1}R_1R_2$ and JR_1R_3 both stabilize $e_1^\perp$:

$$J^{-1}R_1R_2 : \begin{bmatrix} z \\ \frac{\varphi}{\alpha}(z) - \varphi^2 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} \dfrac{\alpha\eta^2\varphi^3 z + \alpha\varphi - i\alpha\eta\varphi^4}{\eta^2\varphi^2 z + -i\alpha\eta - i\eta\varphi^3} \\ \dfrac{\varphi}{\alpha}\left(\dfrac{\alpha\eta^2\varphi^3 z + \alpha\varphi - i\alpha\eta\varphi^4}{\eta^2\varphi^2 z + -i\alpha\eta - i\eta\varphi^3}\right) - \varphi^2 \\ 1 \end{bmatrix}$$

$$JR_1R_3 : \begin{bmatrix} z \\ \frac{\varphi}{\alpha}(z) - \varphi^2 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} \frac{(i\eta\varphi^3 + i\alpha\eta)z - i\eta\alpha\varphi^4 - \alpha\eta^2\varphi}{-\varphi^2 z + \alpha\varphi^3} \\ \frac{\varphi}{\alpha}\left(\frac{(i\eta\varphi^3 + i\alpha\eta)z - i\eta\alpha\varphi^4 - \alpha\eta^2\varphi}{-\varphi^2 z + \alpha\varphi^3}\right) \\ 1 \end{bmatrix}$$

Let A and B be the following elements in $\mathrm{SU}(1,1)$ corresponding to the actions of $J^{-1}R_1R_2$ and JR_1R_3 on $e_1^{\perp}$, respectively.

$$A = \frac{1}{\alpha\sqrt{-i\eta\varphi^3}} \begin{pmatrix} \alpha\eta^2\varphi^3 & \alpha\varphi - i\alpha\eta\varphi^4 \\ \eta^2\varphi^2 & -i\alpha\eta - i\eta\varphi^3 \end{pmatrix}, \qquad B = \frac{1}{\alpha\sqrt{i\eta^3\varphi^3}} \begin{pmatrix} i\eta\varphi^3 + i\alpha\eta & -i\eta\alpha\varphi^4 - \alpha\eta^2\varphi \\ -\varphi^2 & \alpha\varphi^3 \end{pmatrix}.$$

One then sees that

$$\begin{aligned} |\operatorname{Tr}(A)| &= \left|1 + e^{\pi i(t+1/2-1/p)}\right|, \\ |\operatorname{Tr}(B)| &= \left|1 + e^{\pi i(t-1/2+1/p)}\right|, \\ |\operatorname{Tr}(AB)| &= \left|-1 + e^{6\pi i/p}\right|. \end{aligned}$$

All of these values are less than or equal to 2 for admissible values of p and t, so neither A nor B is loxodromic and thus they generate the orientation-preserving subgroup of a Fuchsian triangle group of finite covolume. It follows that Γ_{312} is a lattice in $\mathrm{PU}(1,1)$. By computing orders of these elements for admissible (p,t), one obtains Table 2 showing the corresponding triangle groups, and arithmeticity/non-arithmeticity (A/NA) of each can be checked by comparing with the main theorem of [14]. $\square$

Lemma 7. *Let $K = \langle JR_1R_3,\, JR_2R_1,\, JR_3R_2\rangle$. For all admissible pairs (p,t), the group K is normal in $\tilde{\Gamma}(p,t)$.*

Proof. For indices i,j,k with $k = i+1 \pmod 3$ and $j = i-1 \pmod 3$, the following equations are readily checked:

$$\begin{aligned} R_i(JR_iR_j)R_i^{-1} &= JR_iR_j, \quad & R_k(JR_iR_j)R_k^{-1} &= (JR_iR_j)(JR_jR_k)(JR_iR_j)^{-1}, \\ R_j(JR_iR_j)R_j^{-1} &= JR_kR_i, \quad & J(JR_iR_j)J^{-1} &= JR_kR_i. \end{aligned}$$

$\square$

(p,t)	$\triangle(x,y,z)$	A/NA	(p,t)	$\triangle(x,y,z)$	A/NA
$(3,0)$	$\triangle(2,12,12)$	A	$(4,0)$	$\triangle(4,8,8)$	A
$(3,1/30)$	$\triangle(2,10,15)$	NA	$(4,1/12)$	$\triangle(4,6,12)$	NA
$(3,1/18)$	$\triangle(2,9,18)$	A	$(4,3/20)$	$\triangle(4,5,20)$	NA
$(3,1/12)$	$\triangle(2,8,24)$	NA	$(4,1/4)$	$\triangle(4,4,\infty)$	A
$(3,5/42)$	$\triangle(2,7,42)$	NA	$(4,5/12)$	$\triangle(3,4,12)$	A
$(3,1/6)$	$\triangle(2,6,\infty)$	A	$(5,1/10)$	$\triangle(5,10,10)$	A
$(3,7/30)$	$\triangle(2,5,30)$	A	$(5,1/5)$	$\triangle(4,10,20)$	NA
$(3,1/3)$	$\triangle(2,4,12)$	A	$(5,11/30)$	$\triangle(3,10,30)$	A
			$(5,7/10)$	$\triangle(2,5,10)$	A

Table 3: Properties of Γ_1

Lemma 8. *For each admissible pair (p,t), the group K (as in the previous lemma) has finite index in $\tilde{\Gamma}(p,t)$.*

Proof. $\tilde{\Gamma}(p,t)$ is a quotient of the finitely-presented group

$$\left\langle J,\, R_1,\, R_2,\, R_3 \mid J^3 = R_i^p = \mathrm{Id},\; R_iR_{i+1}R_i = R_{i+1}R_iR_{i+1},\; R_{i+1} = JR_iJ^{-1} \right\rangle$$

where $i = 1,2,3$ (and indices are taken modulo 3). Let $\mathcal{X}_\Gamma$ be some set of additional relations so that $\tilde{\Gamma}(p,t)$ has the presentation

$$\left\langle J,\, R_1,\, R_2,\, R_3 \mid \mathcal{X}_\Gamma,\; J^3 = R_i^p = \mathrm{Id},\; R_iR_{i+1}R_i = R_{i+1}R_iR_{i+1},\; R_{i+1} = JR_iJ^{-1} \right\rangle.$$

As K is normal, we examine the quotient $\tilde{\Gamma}(p,t)/K$ with presentation

$$\left\langle J,\, R_1,\, R_2,\, R_3 \mid \mathcal{X}_\Gamma,\; J^3 = R_i^p = JR_{i+1}R_i = \mathrm{Id},\, R_iR_{i+1}R_i = R_{i+1}R_iR_{i+1},\, R_{i+1} = JR_iJ^{-1} \right\rangle.$$

where, again, $i = 1,2,3$ and the indices are taken modulo 3. Because $\tilde{\Gamma}(p,t)$ is generated by R_1 and J, many of the relations are superfluous, so the presentation for $\tilde{\Gamma}(p,t)/K$ simplifies a bit to

$$\langle J,\, R_1,\, R_2 \mid \mathcal{X}_\Gamma,\; J^3 = R_1^p = JR_2R_1 = \mathrm{Id},\; R_2 = JR_1J^{-1},\; R_1R_2R_1 = R_2R_1R_2\rangle.$$

The relation $JR_2R_1 = \mathrm{Id}$ also makes the braid relation $R_1R_2R_1 = R_2R_1R_2$ redundant, and so the presentation simplifies more to

$$\tilde{\Gamma}(p,t)/K = \left\langle J,\, R_1 \mid \mathcal{X}_\Gamma,\; R_1^p = J^3 = (J^{-1}R_1)^2 = \mathrm{Id} \right\rangle.$$

In this way, one sees that $\tilde{\Gamma}(p,t)/K$ is a quotient of the (orientation-preserving) $(2,3,p)$-triangle group. These triangle groups are finite when $p=3,4,5$, thus K has finite index in $\tilde{\Gamma}(p,t)$. □

Theorem 9. *For each admissible pair (p,t), the hybrid $H := H(\Gamma_1, \Gamma_{312}) = \langle \Lambda_1, \Lambda_{312}\rangle$ has finite index in $\tilde{\Gamma}(p,t)$.*

Proof. From the previous lemma, it suffices to show that the hybrid H contains K. Indeed, H contains the subgroup $\langle J^{-1}R_1R_2, JR_1R_3, R_1, R_3J\rangle$ by Propositions 5 and 6, from which it immediately follows that $JR_1R_3 \in H$. That H contains the other two generators for K is again a straightforward matrix computation.

$$JR_2R_1 = J(J^{-1}R_3J)R_1 = (R_3J)(R_1), \qquad \text{and}$$
$$JR_3R_2 = JR_3(J^{-1}J)R_2(J^{-1}J) = (R_1)(R_3J).$$

□

5 Small phase shift hybrids

When $\tilde{\Gamma}(p,t)$ has small phase shift, we have another natural choice of initial hypersurfaces to use in the hybrid constrution (namely, those coming from a pair of adjacent sides in the core hexagon in Figure 1). In this section, we instead construct hybrids with initial subspaces $v_{312}^{\perp}$ and $v_{321}^{\perp}$ (this choice is essentially unique as each other side is contained in the same J-orbit as one of these two). In homogeneous coordinates, one sees that

$$v_{321}^{\perp} = \{[i\overline{\eta}\varphi, z, 1]^T \ : \ z \in \mathbb{C}\}.$$

Proposition 10. *Γ_{321} is an arithmetic cocompact lattice in $\mathrm{SU}(1,1)$ for all small phase shift values.*

(p,t)	$\triangle(x,y,z)$	A/NA	(p,t)	$\triangle(x,y,z)$	A/NA
$(3,0)$	$\triangle(2,3,12)$	A	$(4,0)$	$\triangle(2,4,8)$	A
$(3,1/30)$	$\triangle(2,3,10)$	A	$(4,1/12)$	$\triangle(2,4,6)$	A
$(3,1/18)$	$\triangle(2,3,9)$	A	$(4,3/20)$	$\triangle(2,4,5)$	A
$(3,1/12)$	$\triangle(2,3,8)$	A	$(5,1/10)$	$\triangle(2,5,5)$	A
$(3,5/42)$	$\triangle(2,3,7)$	A	$(5,1/5)$	$\triangle(2,4,5)$	A

Table 4: Properties of Γ_{321}

Proof. R_2 and JR_3^{-1} both stabilize $v_{321}^{\perp}$:

$$R_2 : \begin{bmatrix} i\overline{\eta}\varphi \\ z \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} i\overline{\eta}\varphi \\ \eta^2 z + \varphi^2 - i\eta\overline{\varphi} \\ 1 \end{bmatrix}$$

$$JR_3^{-1} : \begin{bmatrix} i\overline{\eta}\varphi \\ z \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} i\overline{\eta}\varphi \\ i\overline{\eta}\varphi/z \\ 1 \end{bmatrix}$$

Let A and B be the following elements in $\mathrm{SU}(1,1)$ corresponding to the actions of R_2 and JR_3^{-1} on $v_{321}^{\perp}$, respectively.

$$A = \frac{1}{\eta}\begin{pmatrix} \eta^2 & \varphi^2 - i\eta\overline{\varphi} \\ 0 & 1 \end{pmatrix}, \qquad B = \frac{1}{\sqrt{-i\overline{\eta}\varphi}}\begin{pmatrix} 0 & i\overline{\eta}\varphi \\ 1 & 0 \end{pmatrix}.$$

One can check that

$$\begin{aligned} |\operatorname{Tr}(A)| &= |e^{i\pi/p} + e^{-i\pi/p}|, \\ |\operatorname{Tr}(B)| &= 0, \\ |\operatorname{Tr}(A^{-1}B)| &= |e^{i\pi(1/2+1/p-t/3)} - e^{2\pi i t/3}|. \end{aligned}$$

All of these values are less than 2 when $p \geq 3$ and $|t| \neq \frac{1}{2} - \frac{1}{p}$ and so the elements are elliptic. Thus $\langle A, B\rangle$ is a cocompact triangle group (and therefore Γ_{321} is a cocompact lattice). By computing orders of these elements for (p,t) values in Table 1, one obtains Table 4 showing the corresponding triangle groups, and arithmeticity/non-arithmeticity (A/NA) of each can be checked by comparing with the main theorem of [14]. □

Theorem 11. *For $|t| < \frac{1}{2} - \frac{1}{p}$, the hybrid $H(\Gamma_{312}, \Gamma_{321}) := \langle \Lambda_{312}, \Lambda_{321}\rangle$ is the full lattice $\tilde{\Gamma}(p,t)$.*

Proof. The group $K = \langle R_1, R_3J, R_2, JR_3^{-1}\rangle$ is a subgroup of $H(\Gamma_{312}, \Gamma_{321})$. Since $J = (R_3J)^{-1}(JR_3^{-1})^{-1}$, $K = \langle R_1, J\rangle = \tilde{\Gamma}(p,t)$. □

By comparing with the table on Page 418 of [7], one sees that Γ_{312} and Γ_{321} are both arithmetic and noncommensurable in the cases where $(p,t) = (3, 1/12), (4, 1/12)$ and $(5, 1/5)$. Thus

Corollary 12. *$\tilde{\Gamma}(3, 1/12)$, $\tilde{\Gamma}(4, 1/12)$ and $\tilde{\Gamma}(5, 1/5)$ are non-arithmetic lattices obtained by interbreeding two noncommensurable arithmetic lattices.*

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